2025 京都大(理系)

$\boxed{1}$

問1.

$|z|^2 = z\bar{z} = 4$

$\therefore \frac{1}{z} = \frac{\bar{z}}{4}$

$\left|z - \frac{i}{z}\right|^2$ $\quad z = 2(\cos\theta + i\sin\theta)$

$= \left|z - \frac{i}{4}\bar{z}\right|^2$

$= \left|2\cos\theta + 2i\sin\theta - \frac{i}{4}(2\cos\theta - 2i\sin\theta)\right|^2$

$= \left|2\cos\theta - \frac{1}{2}\sin\theta + i\left(2\sin\theta - \frac{1}{2}\cos\theta\right)\right|^2$

$= \left(2\cos\theta - \frac{1}{2}\sin\theta\right)^2 + \left(2\sin\theta - \frac{1}{2}\cos\theta\right)^2$

$= 4 + \frac{1}{4} - 4\sin\theta\cos\theta$

$= \frac{17}{4} - 2\sin 2\theta$

$\left|z - \frac{i}{z}\right|$の最大値 $\sqrt{\frac{25}{4}} = \frac{5}{2}$

〃 最小値 $\sqrt{\frac{9}{4}} = \frac{3}{2}$ //

問2. (1)

$\int_0^{\sqrt{3}} \frac{x\sqrt{x^2+1}}{x^2+1}\,dx$

$= \int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$

$t = x^2+1$ とおく

$dt = 2x\,dx$

$= \int_1^4 \frac{1}{\sqrt{t}} \cdot \frac{1}{2}\,dt$

$= \left[t^{\frac{1}{2}}\right]_1^4$

$= 1$

$$\begin{array}{r|rrrr} & & & 2 & \\ 1\ 0\ 1 & 2 & 0 & 0 & 1 \\ & 2 & 0 & 2 & \\ \hline & & & -2 & 1 \end{array}$$

$\int_0^{\sqrt{3}} \frac{2x^3+1}{x^2+1}\,dx$

$= \int_0^{\sqrt{3}} \left(2x + \frac{-2x+1}{x^2+1}\right)dx$

$= \left[x^2 - \log|x^2+1|\right]_0^{\sqrt{3}} + \left[\arctan x\right]_0^{\sqrt{3}}$

$= 3 - \log 4 + \frac{\pi}{3}$

(与式)

$= 1 + 3 - 2\log 2 + \frac{\pi}{3}$

$= 4 + \frac{\pi}{3} - 2\log 2$ //

(2)

$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos x}{1+\cos x}}\,dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}\,dx$

$= \int_0^{\frac{\pi}{2}} \left|\tan\frac{x}{2}\right|dx$

$= \int_0^{\frac{\pi}{2}} \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx$

$= \left[-2\log\left|\cos\frac{x}{2}\right|\right]_0^{\frac{\pi}{2}}$

$= -2\log\frac{1}{\sqrt{2}}$

$= 2\log\sqrt{2} = \log 2$ //

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以下3を法とする合同式とする.

$x \equiv 0 \Rightarrow x^6 \equiv 0$

$x \equiv \pm 1 \Rightarrow x^6 \equiv 1$

$y \equiv 0 \Rightarrow y^4 \equiv 0$

$y \equiv \pm 1 \Rightarrow y^4 \equiv 1$

$x^6 + y^4 = 9z^2$ より $x^6 + y^4$ は9の倍数つまり3の倍数だがその

ためには $x \equiv 0$ かつ $y \equiv 0$ となるしかない.

$x = 3k,\ y = 3l\ (k \in \mathbb{N},\ l \in \mathbb{N})$

とおくと

$9z^2 = 3^6k^6 + 3^4l^4$

$z^2 = 81k^6 + 9l^4$

右辺が3の倍数より z^2 と z も3の倍数.

$z = 3m\ (m \in \mathbb{N})$

とおくと

$9m^2 = 81k^6 + 9l^4$

$m^2 = 9k^6 + l^4$

$k=1$ のとき $l=1$ は不適.

$l=2$ で $m=5$.

$l=1$ のとき $k \geqq 2$ だと

$m^2 \geqq 9 \cdot 2^6 + 1 \geqq 24^2$

となり $m \geqq 24$ となって最小にならない. つまり求める最小値は

$N = 9 \cdot 15^2 = 2025$ //

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$f(x)=x^2\log x$

$f'(x)=2x\log x+x$

$l_t: y=-\dfrac{1}{t(2\log t+1)}(x-t)$
$\qquad +t^2\log t-\dfrac{1}{1+2\log t}$

$\downarrow (P(t),0)$

$0=-\dfrac{1}{t(2\log t+1)}\{P(t)-t\}$
$\qquad +t^2\log t-\dfrac{1}{1+2\log t}$

$P(t)-t=t^3(1+2\log t)\log t-t$

$P(t)=t^3\{2(\log t)^2+\log t\}$

$P'(t)=3t^2\{2(\log t)^2+\log t\}$
$\qquad +t^3\left\{\dfrac{4}{t}\log t+\dfrac{1}{t}\right\}$
$\quad =t^2\{6(\log t)^2+7\log t+1\}$
$\quad =t^2(6\log t+1)(\log t+1)$

t	$\frac{1}{\sqrt{e}}$ … $e^{-\frac{1}{6}}$ … e
$P'(t)$	$-$ 0 $+$
$P(t)$	0 $\searrow$ $\nearrow 3e^3$

$-\dfrac{1}{9\sqrt{e}}\leqq P(t)\leqq 3e^2$

$\boxed{4}$

(1)

$\overrightarrow{OQ}=\dfrac{1}{4}\overrightarrow{OA}+\dfrac{1}{2}\overrightarrow{OB}+\dfrac{3}{4}\overrightarrow{OC}$ とおく.

$\quad =\dfrac{1}{4s}\overrightarrow{OL}+\dfrac{1}{2t}\overrightarrow{OM}+\dfrac{3}{4u}\overrightarrow{ON}$

$\dfrac{1}{s}+\dfrac{2}{t}+\dfrac{3}{u}=4\iff\dfrac{1}{4s}+\dfrac{1}{2t}+\dfrac{3}{4u}=1$

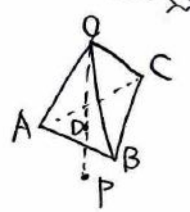
よりQはs, t, uに関わらず平面LMN上にある. このQが題意のPである.

(2)

$\overrightarrow{OP}=\dfrac{1}{4}\overrightarrow{OA}+\dfrac{1}{2}\overrightarrow{OB}+\dfrac{3}{4}\overrightarrow{OC}$

$\quad =\dfrac{3}{2}\underbrace{\left(\dfrac{1}{6}\overrightarrow{OA}+\dfrac{1}{3}\overrightarrow{OB}+\dfrac{1}{2}\overrightarrow{OC}\right)}$

$\overrightarrow{OD}$ とおく.



求める体積は

$\dfrac{1}{2}V$

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$\overrightarrow{AP}=\begin{pmatrix}\cos\theta\\ \sin\theta\\ \frac{1}{2}\cos\theta-\frac{\sqrt{2}}{4}\end{pmatrix}$

直線AP: $\begin{cases}x=(\cos\theta)t\\ y=(\sin\theta)t\\ z=\frac{\sqrt{2}}{4}+\left(\frac{1}{2}\cos\theta-\frac{\sqrt{2}}{4}\right)t\end{cases}$

$\downarrow z=0$

$\sqrt{2}+(2\cos\theta-\sqrt{2})t=0$

$\iff t=\dfrac{1}{1-\sqrt{2}\cos\theta}$

$Q(X,Y,0)$ とおくと

$\begin{cases}X=\dfrac{\cos\theta}{1-\sqrt{2}\cos\theta}\\ Y=\dfrac{\sin\theta}{1-\sqrt{2}\cos\theta}\end{cases}$

$(1-\sqrt{2}\cos\theta)X=\cos\theta$

$X=(\sqrt{2}X+1)\cos\theta$

$\cos\theta=\dfrac{X}{\sqrt{2}X+1}$

$\left(X=-\dfrac{1}{\sqrt{2}}\right.$ とおくと不適. $\left.\therefore X\neq-\dfrac{1}{\sqrt{2}}\right)$

$(1-\sqrt{2}\cos\theta)Y=\sin\theta$

$\downarrow$ 2乗

$\left(1-\dfrac{\sqrt{2}X}{\sqrt{2}X+1}\right)^2Y^2=1-\left(\dfrac{X}{\sqrt{2}X+1}\right)^2$

$Y^2=(\sqrt{2}X+1)^2-X^2$
$\quad =X^2+2\sqrt{2}X+1$
$\quad =(X+\sqrt{2})^2-1$

$\therefore (X+\sqrt{2})^2-Y^2=1$

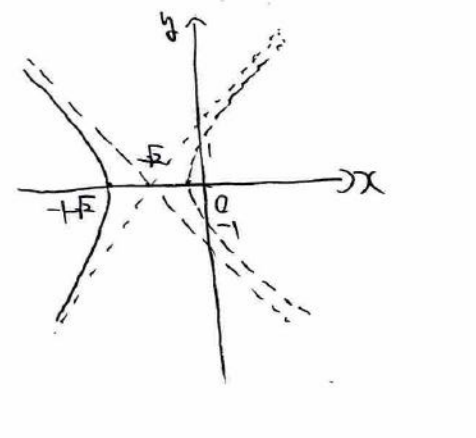
$X=\dfrac{1}{\frac{1}{\cos\theta}-\sqrt{2}}$

$-\dfrac{\pi}{4}<\theta<\dfrac{\pi}{4}$ より

$X\leqq\dfrac{1}{1-\sqrt{2}}=-1-\sqrt{2}$

求めるQの軌跡は

双曲線 $(x+\sqrt{2})^2-y^2=1$ $(x\leqq-1-\sqrt{2})$



$\boxed{6}$

$Y_n = X_1X_2 + X_2X_3 + \cdots + X_{n-1}X_n$

$a_n = P(Y_n$ が奇数 $\cap X_n = 0)$

$b_n = P(Y_n$ が偶数 $\cap X_n = 0)$

$c_n = P(Y_n$ が奇数 $\cap X_n = 1)$

$d_n = P(Y_n$ が偶数 $\cap X_n = 1)$

とおく. $P_n = a_n + c_n$.

$$\begin{cases} a_{n+1} = \frac{1}{2}a_n + \frac{1}{2}c_n \\ b_{n+1} = \frac{1}{2}b_n + \frac{1}{2}d_n \\ c_{n+1} = \frac{1}{2}a_n + \frac{1}{2}d_n \\ d_{n+1} = \frac{1}{2}b_n + \frac{1}{2}c_n \end{cases}$$

$$a_{n+2} = \frac{1}{2}a_{n+1} + \frac{1}{2}c_{n+1} = \frac{1}{2}a_n + \frac{1}{4}(c_n + d_n)$$

$$b_{n+2} = \frac{1}{2}b_{n+1} + \frac{1}{2}d_{n+1} = \frac{1}{2}b_n + \frac{1}{4}(c_n + d_n)$$

$$c_{n+2} = \frac{1}{2}a_{n+1} + \frac{1}{2}d_{n+1} = \frac{1}{2}c_n + \frac{1}{4}(a_n + b_n)$$

$$d_{n+2} = \frac{1}{2}d_n + \frac{1}{4}(a_n + b_n)$$

$$a_{n+2} + c_{n+2} - b_{n+2} - d_{n+2} = \frac{1}{2}(a_n + c_n - b_n - d_n)$$

$$P_{n+2} - (1 - P_{n+2}) = \frac{1}{2}(P_n - (1 - P_n))$$

整理して

$$P_{n+2} = \frac{1}{2}P_n + \frac{1}{4}$$

$$\Leftrightarrow P_{n+2} - \frac{1}{2} = \frac{1}{2}\left(P_n - \frac{1}{2}\right)$$

$$P_{2m} - \frac{1}{2} = \left(\frac{1}{2}\right)^{m-1}\left(P_2 - \frac{1}{2}\right)$$

$$\therefore P_{2m} = \frac{1}{2} - \left(\frac{1}{2}\right)^{m+1}$$

$$P_{2m+1} - \frac{1}{2} = \left(\frac{1}{2}\right)^{m-1}\left(P_3 - \frac{1}{2}\right)$$

$$\therefore P_{2m+1} = \frac{1}{2} - \left(\frac{1}{2}\right)^{m+1}$$

以上より

$$P_n = \begin{cases} \frac{1}{2} - \left(\frac{1}{2}\right)^{\frac{n}{2}+1} & (n\text{は偶数}) \\ \frac{1}{2} - \left(\frac{1}{2}\right)^{\frac{n+1}{2}} & (n\text{は奇数}) \end{cases}$$

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