2024 慶應義塾大（理工）

1.

(1) $2024 = 2^3 \cdot 11 \cdot 23$

約数は $4 \cdot 2 \cdot 2 = 16$コ.

1	2024
2	1012
4	506
8	253
11	184
22	92
⋮	

$2024^{\frac{1}{6}} = \sqrt{2} \cdot 253^{\frac{1}{6}}$

$2 < 253^{\frac{1}{6}} < 3$

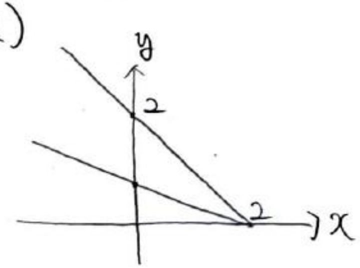
$\Leftrightarrow 2\sqrt{2} < 2024^{\frac{1}{6}} < 3\sqrt{2}$

$(3.5)^6 = (42.875)^2$
$< 43^2 = 1849$

$\therefore 3.5 < 2024^{\frac{1}{6}} < 3\sqrt{2}$

4が1番近い

(2)



$a_n \leqq 2$において

$\frac{2}{3} - \frac{1}{3}a_n \leqq f(a_n) \leqq 2 - a_n$

$\Leftrightarrow \frac{2}{3} + \frac{2}{3}a_n \leqq a_{n+1} \leqq 2 \cdots \cancel{*}$

ここで $a_1 \leqq 2 \Rightarrow a_1 \leqq a_n \leqq 2$

となることを数学的帰納法で示す.

(i) $n=1$のとき成立.

(ii) $n=k$のとき成立つとする.

$*$より $n=k+1$のとき

$\frac{2}{3} + \frac{2}{3}a_k \leqq a_{k+1} \leqq 2$

ここで

$a_{k+1} - a_1$

$\geqq \frac{2}{3} + \frac{2}{3}a_k - a_1$

$\geqq \frac{2}{3} + \frac{2}{3}a_1 - a_1$

$\geqq \frac{2}{3} - \frac{1}{3}a_1 \geqq 0 \quad (\because a_1 \leqq 2)$

$\therefore a_1 \leqq a_{k+1}$

以上より

$a_1 \leqq 2 \Rightarrow a_1 \leqq a_{k+1} \leqq 2$

このときも成立.

(i)(ii)よりすべての自然数nにおいて成立.

(2) $*$より

$a_{n+1} - 2 \geqq \frac{2}{3}a_n - \frac{4}{3}$

$= \frac{2}{3}(a_n - 2)$

これより

$a_n - 2 \geqq (a_1 - 2)\left(\frac{2}{3}\right)^{n-1}$

$\therefore (a_1 - 2)\left(\frac{2}{3}\right)^{n-1} \leqq a_n - 2 \leqq 0$

$\lim_{n\to\infty} (a_1 - 2)\left(\frac{2}{3}\right)^{n-1} = 0$ より

$\lim_{n\to\infty} (a_n - 2) = 0$

$\therefore \lim_{n\to\infty} a_n = \underline{2}$

2.

(1) $\frac{2+3}{12} = \underline{\frac{5}{12}}$

(2) $\frac{3}{5}$

(3) $\frac{\frac{2}{6} \cdot 1}{\frac{1}{6} \cdot 0 + \frac{2}{6} \cdot 1 + \frac{3}{6}\left(\frac{1}{2}\right)^2} = \underline{\frac{8}{11}}$

(4) $\frac{3}{6} \cdot \frac{2}{4} = \underline{\frac{1}{4}}$

(5) $\frac{1}{6} \cdot 1 + \frac{2}{6} \cdot 1 + \frac{3}{6}\left\{\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n\right\}$

$= \underline{\frac{1}{2} + \left(\frac{1}{2}\right)^n}$

3

(1) $1 \leqq x \leqq 3$で $\frac{f(x)}{x}$は単調減少.

$S = \int_1^3 |f(x) - ax| dx$

$= \int_1^3 x\left|\frac{f(x)}{x} - a\right| dx$

$= \begin{cases} \int_1^3 \{f(x) - ax\} dx & \left(a \leqq \frac{f(3)}{3}\right) \\ \int_1^3 \{ax - f(x)\} dx & (a \geqq f(1)) \end{cases}$

$= \begin{cases} \underline{I - 4a} & \left(a \leqq \frac{f(3)}{3}\right) \\ 4a - I & (a \geqq f(1)) \end{cases}$

(2)

$1 < x < 3$で $\frac{f(x)}{x}$は単調減少.

$g(x) = \frac{f(x)}{x} - a$ とおくと.

$g(1) > 0$, $g(3) < 0$

$1 < x < 3$で $g(x) = 0$ つまり

$f(x) - ax = 0$は解をただ1つもつ.

(3) $g(t) = \frac{f(t)}{t} - a = 0$

$\therefore a = \underline{\frac{f(t)}{t}}$

S

$= \int_1^t \{f(x) - ax\} dx + \int_t^3 \{ax - f(x)\} dx$

$= \int_1^t f(x) dx - a\cdot\frac{t^2-1}{2} + a\cdot\frac{9-t^2}{2} + \int_3^t f(x) dx$

$= F(t) - at^2 + 5a + F(t) + \int_3^1 f(x)dx$

$= 2F(t) - \frac{f(t)}{t}(t^2-5) - F(3)$

$\therefore g(t) = \frac{5-t^2}{t}$ //

(4) 平均値の定理を使う.

(i) $1 \leqq x < t_0$ のとき

$\begin{cases} \frac{F(x)-F(t_0)}{x-t_0} = F'(c) = f(c) \\ x < c < t_0 \end{cases}$

をみたす c が存在.

$\frac{F(x)-F(t_0)}{x-t_0} = f(c) < f(x)$

$\therefore F(x) - F(t_0) > (x-t_0) f(x)$

(ii) $t_0 < x < 3$ のとき

$\begin{cases} \frac{F(x)-t_0}{x-t_0} = f(c) \\ t_0 < c < x \end{cases}$

をみたす c が存在.

$\frac{F(x)-F(t_0)}{x-t_0} = f(c) > f(x)$

$\therefore F(x) - F(t_0) > (x-t_0) f(x)$

(iii) $x = t_0$ のとき

$F(x) - F(t_0) = (x-t_0) f(x)$

(i)~(iii)より

$F(x) - F(t_0) \geqq (x-t_0) f(x)$

(5) $f(x) > 0$ より

$\underbrace{2(x-t_0) + p(x)}_{h(x)} \geqq 0$

$h(t_0) = 0$

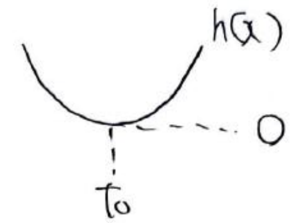
$h'(x) = 2 + p'(x)$

$h''(x) = p''(x) > 0 \to h'(x)$ は単調増加.

$h'(t_0) = 2 + p'(t_0) = 0$

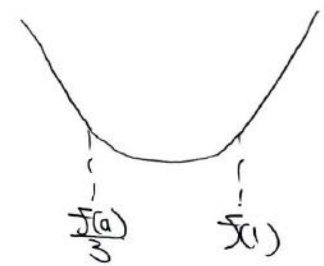
$\therefore p'(t_0) = -2$

であれば



$h(x) \geqq 0$ となる.

(6)



$g(x) = \frac{5-x^2}{x}$

$g'(x) = -\frac{5}{x^2} - 1$

$g''(x) = \frac{10}{x^3}$

$t_0 = \sqrt{5}$ として

$F(t) - F(\sqrt{5}) \geqq (t-\sqrt{5}) f(t)$

$\Longleftrightarrow 2F(t) \geqq 2F(\sqrt{5}) + 2(t-\sqrt{5}) f(t)$

$p(x) = g(x)$ として

$2(t-\sqrt{5}) f(t) + g(t) f(t) \geqq 0$

S

$\geqq 2F(\sqrt{5}) + 2(t-\sqrt{5}) f(t) - F(3) + g(t) f(t)$

$\geqq 2F(\sqrt{5}) - F(3)$

$t = \sqrt{5}$ で等号成立 このとき

S は最小. $a = \frac{f(\sqrt{5})}{\sqrt{5}}$ //

4.

(1) $\triangle OAB = \frac{1}{2}\sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$

$= \frac{\sqrt{3}}{2}$ //

$\overrightarrow{OH} = s\vec{a} + t\vec{b}$ とおく.

$\overrightarrow{CH} = s\vec{a} + t\vec{b} - \vec{c}$ で

$\begin{cases} \overrightarrow{CH}\cdot\vec{a} = s + t + 1 = 0 \\ \overrightarrow{CH}\cdot\vec{b} = s + 4t = 0 \end{cases}$

$\Rightarrow s = -\frac{4}{3}, t = \frac{1}{3}$

$\therefore \overrightarrow{CH} = -\frac{4}{3}\vec{a} + \frac{1}{3}\vec{b} - \vec{c}$

$|\overrightarrow{CH}|^2 = \cdots = \frac{8}{3}$

(体積) $= 2\cdot\triangle OAB\cdot\frac{2\sqrt{2}}{\sqrt{3}}$

$= \frac{\sqrt{2}}{3}$ //

(2)

$\overrightarrow{JI}\cdot\overrightarrow{JK}$

$=(\overrightarrow{OI}-\overrightarrow{OJ})\cdot(\overrightarrow{OK}-\overrightarrow{OJ})$

$=(t\vec{a}-\frac{1}{2}\vec{b})\cdot(\vec{b}+\frac{1}{2}\vec{c}-\frac{1}{2}\vec{b})$

$=\cdots$

$=-1$ //

$|\overrightarrow{JI}|^2=t^2-t+1$

$|\overrightarrow{JK}|^2=2$

$\triangle IJK=\frac{1}{2}\sqrt{|\overrightarrow{JI}|^2|\overrightarrow{JK}|^2-(-1)^2}$

$=\frac{1}{2}\sqrt{2t^2-2t+1}$ //

(3) DEとの交点をPとする。

$\overrightarrow{OP}=\overrightarrow{OJ}+\overrightarrow{JP}$

$=\overrightarrow{OJ}+\alpha\overrightarrow{JI}+\beta\overrightarrow{JK}$

$=\frac{1}{2}\vec{b}+\alpha(t\vec{a}-\frac{1}{2}\vec{b})+\beta(\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c})$

また

$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AD}+\overrightarrow{DP}$

$=\vec{a}+\vec{c}+\gamma\vec{b}$

これらが一致するので

$\begin{cases} t\alpha=1 \\ \frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}\beta=\gamma \\ \frac{\beta}{2}=1 \end{cases} \rightarrow \begin{matrix}\alpha=\frac{1}{t} \\ \beta=2\end{matrix}$

$\therefore \gamma=\frac{1}{2}(3-\frac{1}{t})$

$\downarrow\ 0\leqq\gamma\leqq 1$

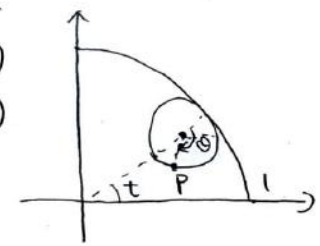
$0\leqq\frac{1}{2}(3-\frac{1}{t})\leqq 1$

解と $\frac{1}{3}\leqq t<1$ $(\because 0<t<1)$ //

5.

(1)

(2)



$2\pi\cdot\frac{t}{2\pi}=2r\pi\cdot\frac{\theta}{2\pi}$

$\theta=\frac{t}{r}$ $\rightarrow t=2\pi r$ //

$w(t)=(1-r)(\cos t+i\sin t)$

$z(t)=w(t)$

$+r(\cos(t-\frac{t}{r})$

$+i\sin(t-\frac{t}{r}))$ //

$x=(1-r)\cos t+r\cos(1-\frac{1}{r})t$

$y=(1-r)\sin t+r\sin(1-\frac{1}{r})t$

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$

$=\cdots$

$=(1-r)^2 2(1-\cos\frac{t}{r})$

L

$=\int_0^{2\pi r}\sqrt{2(1-r)^2(1-\cos\frac{t}{r})}\,dt$

$=\int_0^{2\pi r}2(1-r)\sin\frac{t}{2r}\,dt$

$=\cdots$

$=8r(1-r)$ //

(4)

$Ln\leqq l(t)<L(n+1)$

$\therefore \frac{Ln}{2\pi r(n+1)}<\frac{l(t)}{t}<\frac{L(n+1)}{2\pi rn}$

$\lim_{t\to\infty}\frac{l(t)}{t}=\frac{L}{2\pi r}=\frac{4(1-r)}{\pi}$ //