2023 慶應理工

1.

(1) $f(x)=x^4$

$f'(a)$

$=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$=\lim_{h\to 0}\frac{(a+h)^4-a^4}{h}$

$=\lim_{h\to 0}\frac{4a^3h+6a^2h^2+4ah^3+h^4}{h}$

$=\lim_{h\to 0}(4a^3+6a^2h+4ah^2+h^3)$

$=4a^3$

(2)

$\lim_{h\to +0}\frac{g(0+h)-g(0)}{h}$

$=\lim_{h\to +0}\frac{h\sqrt{h^2+1}-0}{h}$

$=\lim_{h\to +0}\sqrt{h^2+1}=1$

$\lim_{h\to -0}\frac{g(0+h)-g(0)}{h}$

$=\lim_{h\to -0}\frac{-h\sqrt{h^2+1}-0}{h}$

$=\lim_{h\to -0}(-\sqrt{h^2+1})=-1$

これより $g(x)$ の $x=0$ での微分係数が存在しない.

(3)

平均値の定理より $0\leq a<b\leq 1$ で

$\begin{cases}\frac{h(b)-h(a)}{b-a}=h'(c) \cdots ① \\ a<c<b\end{cases}$

なる c が存在. ①より

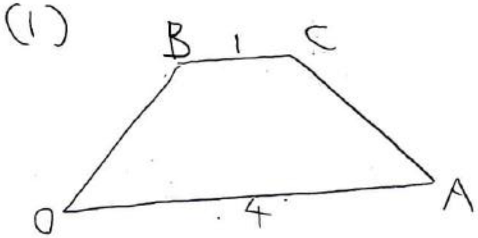
$h(b)-h(a)=h'(c)<0$

$\therefore h(a)>h(b)$

これより $h(x)$ は区間 $[0,1]$ で減少する.

2.

(1)



$\cos\angle AOB=\frac{28k-20k+8k}{8k\cdot\sqrt{76}}$

$=\frac{2}{\sqrt{76}}=\frac{1}{\sqrt{19}}$ //

(台形OACB)

$=\frac{5}{4}\triangle AOB$

$=\frac{5}{4}\cdot\frac{1}{2}\cdot 8k\cdot 2\sqrt{19}\cdot\frac{3\sqrt{2}}{\sqrt{19}}$

$=5k\cdot 2\cdot 3\sqrt{2}=30\sqrt{2}k$ //

$\overrightarrow{BC}=\frac{1}{4}\overrightarrow{OA}$

$\overrightarrow{AC}-\overrightarrow{AB}$

$\Leftrightarrow \overrightarrow{AC}=\overrightarrow{AB}+\frac{1}{4}\overrightarrow{OA}$

$\Leftrightarrow \overrightarrow{AC}=\overrightarrow{OB}-\frac{3}{4}\overrightarrow{OA}$

$=\begin{pmatrix}7-3k\\5+3k\\-\sqrt{2}+3\sqrt{2}k\end{pmatrix}$

$|\overrightarrow{AC}|^2$

$=49-42k+9k^2+25+30k+9k^2+18k^2-12k+2$

$=36k^2-24k+76$

$|\overrightarrow{AC}|$

$=2\sqrt{9k^2-6k+19}$ //

(2)

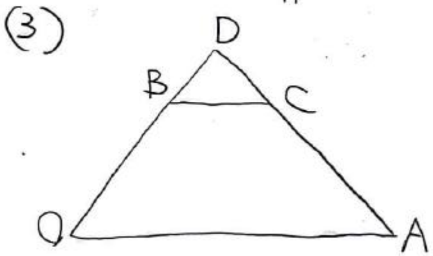
円に内接するとき台形OACBは等脚台形になる.

$|\overrightarrow{AC}|^2=76$

$\Leftrightarrow 3k^2-2k=0$

$\therefore k=\frac{2}{3}$ //

(3)



$\triangle OBP=\triangle ACP$

$\Leftrightarrow \triangle ODP=\triangle ADP$

$\Leftrightarrow OD\cdot DP\sin\angle ODP=AD\cdot DP\sin\angle ADP$

$\Leftrightarrow \sin\angle ODP=\sin\angle ADP$

$\Leftrightarrow \cos\angle ODP = \pm\cos\angle ADP$

$\Leftrightarrow \dfrac{\overrightarrow{DO}\cdot\overrightarrow{DP}}{|\overrightarrow{DO}||\overrightarrow{DP}|} = \pm\dfrac{\overrightarrow{DA}\cdot\overrightarrow{DP}}{|\overrightarrow{DA}|\cdot|\overrightarrow{DP}|}$

$\Leftrightarrow \overrightarrow{DO}\cdot\overrightarrow{DP} = \pm\overrightarrow{DA}\cdot\overrightarrow{DP}$

(i) $\overrightarrow{DO}\cdot\overrightarrow{DP} = \overrightarrow{DA}\cdot\overrightarrow{DP}$ のとき

$\overrightarrow{DO}\cdot\overrightarrow{DP} = \overrightarrow{DA}\cdot\overrightarrow{DP}$

$\Leftrightarrow \underbrace{(\overrightarrow{DO}-\overrightarrow{DA})}_{\overrightarrow{AO}}\cdot\overrightarrow{DP} = 0$

これはDを通りOAに垂直な平面となるのでOAと交わる.

(ii) $\overrightarrow{DO}\cdot\overrightarrow{DP} = -\overrightarrow{DA}\cdot\overrightarrow{DP}$ のとき

$(\overrightarrow{DO}+\overrightarrow{DA})\cdot\overrightarrow{DP} = 0$

ここで $\overrightarrow{OD} = \frac{4}{3}\overrightarrow{OB}$

$= \begin{pmatrix} \frac{28}{3} \\ \frac{20}{3} \\ -\frac{4}{3}\sqrt{2} \end{pmatrix}$

$P(x, y, z)$ とすると

$(\overrightarrow{OA} - 2\overrightarrow{OD})\cdot(\overrightarrow{OP} - \overrightarrow{OD}) = 0$

$\Leftrightarrow \begin{pmatrix} -16 \\ -16 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} x-\frac{28}{3} \\ y-\frac{20}{3} \\ z+\frac{4}{3}\sqrt{2} \end{pmatrix} = 0$

$\therefore x + y - 16 = 0$

線分OA上の点 $(t, -t, -\sqrt{2}t)$ は上の平面上にない。

つまり上が α.

Oと α の距離は

$\dfrac{|0+0-16|}{\sqrt{2}} = 8\sqrt{2}$ //

点と平面の距離

3.

(1) 4回のとき

$P(A=4)$

$= \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$P(A=3)$

$= P(HHHT)$
$+ P(HHTH)$
$+ P(TTHH)$
$+ P(THHH)$
$+ P(HTHH)$
$+ P(TTTH)$

$= \frac{1}{16}\times 6$

$\therefore P(A\geqq 3) = \frac{7}{16}$ //

7回で

$P(A=4 \cap B\leqq 3)$
$= \frac{1}{16}\cdot\left(\frac{1}{2}\right)^3$

$P(HHHT \cap B\leqq 3)$
$= \frac{1}{16}\cdot\left(\frac{1}{2}\right)^3$

$P(HHTH \cap B\leqq 3)$
$= \frac{1}{16}\cdot P\begin{pmatrix} HHH \\ THH \end{pmatrix}$
$= \frac{1}{16}\cdot\left(\frac{1}{2}\right)^3\cdot 2$

$P(TTHH \cap B\leqq 3)$
$= \frac{1}{16}\cdot\left(\frac{1}{2}\right)^3\cdot 2$

$P(THHH \cap B\leqq 3)$
$= \frac{1}{16}\cdot\left(\frac{1}{2}\right)^3\cdot 2$

$P(HTHH \cap B\leqq 3)$
$= \frac{1}{16}\cdot P\begin{pmatrix} HHH \\ THH \end{pmatrix}$
$= \frac{1}{16}\left(\frac{1}{2}\right)^3 2$

$P(TTTH \cap B\leqq 3)$
$= \frac{1}{16}\cdot\left(\frac{1}{2}\right)^3$

求める条件付き確率は

$\dfrac{\frac{1}{16}\cdot\frac{1}{8}(1+1+2+2+2+2+1)}{\frac{7}{16}}$

$= \frac{11}{56}$ //

(2)

n回目 $\frac{1}{2}$ n+1回目

Aの方が多い P_n → $\frac{1}{2}$ P_{n+1}

同じ $1-2P_n$ → $1-2P_{n+1}$

Bの方が多い P_n → $\frac{1}{2}$

$P_{n+1} = \frac{1}{2}P_n + \frac{1}{2}(1-2P_n)$

$= -\frac{1}{2}P_n + \frac{1}{2}$ //

↓

$P_n = C\left(-\frac{1}{2}\right)^{n-1} + \frac{1}{3}$

↓ $n=1$

$P_1 = C + \frac{1}{3} = \frac{1}{2}$ $\therefore C = \frac{1}{6}$

$\therefore P_n = \frac{1}{6}\left(-\frac{1}{2}\right)^{n-1} + \frac{1}{3}$ //

(3) n回目のとき ($n \geq 4$)

$P(A \geq n-1)$

$= \left(\frac{1}{2}\right)^n + P(A = n-1)$

$= \left(\frac{1}{2}\right)^n$

$+ P(HH \sim HT)$

$+ P(H \times n-2, T \times 1 \to H)$

$+ P(TTHH \sim H)$

$+ P(TTTH \sim H)$

$\vdots$

$+ P(TT \cdots TH)$

$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n + (n-1)\left(\frac{1}{2}\right)^n$

$+ (n-2)\left(\frac{1}{2}\right)^n$

$= (2n-1)\left(\frac{1}{2}\right)^n$ //

$u_n = P(A = n-2)$ とおく

$\Downarrow$

u_{n+1}

$= \frac{1}{2}u_n + P(A = n-1)$

$- 2\left(\frac{1}{2}\right)^{n+1} - 2\left(\frac{1}{2}\right)^{n+1}$

$= \frac{1}{2}(u_n + 2(n-2)\left(\frac{1}{2}\right)^n$

$\Leftrightarrow 2^{n+1}u_{n+1} = 2^n u_n + 4(n-2)$

($2^{n+1}u_{n+1} = b_{n+1}$, $2^n u_n = b_n$)

$b_n = b_4 + \sum_{k=4}^{n-1} 4(k-2)$

$= \cdots$

$= 2(n-1)(n-4)$

$\therefore u_n = (2n^2 - 10n + 14)\left(\frac{1}{2}\right)^n$

$\therefore P(A \geq n-2)$

$= (2n-1)\left(\frac{1}{2}\right)^n + (2n^2 - 10n + 14)\left(\frac{1}{2}\right)^n$

$= (2n^2 - 8n + 13)\left(\frac{1}{2}\right)^n$ //

4.

(1) $|b| \leqq |(1-\cos x)b+1|$

(i) $b \geqq 0$ のとき

$b \leqq (1-\cos x)b+1$

$\Leftrightarrow (\cos x)b \leqq 1$

$\Leftrightarrow b \leqq \dfrac{1}{\cos x}$

(ii) $-\dfrac{1}{1-\cos x} \leqq b < 0$ のとき

$-b \leqq (1-\cos x)b+1$

$\Leftrightarrow -1 \leqq (2-\cos x)b$

$\Leftrightarrow b \geqq -\dfrac{1}{2-\cos x}$

(iii) $b < -\dfrac{1}{1-\cos x}$ のとき

$-b \leqq -(1-\cos x)b-1$

$\Leftrightarrow 1 \leqq (\cos x)b$

$\Leftrightarrow b \geqq \dfrac{1}{\cos x}$ …不適

以上より

$-\dfrac{1}{2-\cos x} \leqq b \leqq \dfrac{1}{\cos x}$

常に成り立つには

$-\dfrac{1}{2} \leqq b \leqq 1$ //

(2)

$\left|\dfrac{b}{b+1-\cos x}\right| \leqq 1$

$\Leftrightarrow -1 \leqq \dfrac{b}{b+1-\cos x} \leqq 1$

$\therefore -1 \leqq \left(\dfrac{b}{b+1-\cos x}\right)^n \leqq 1$

$\therefore -\sin x(\cos x)^{n-1}$

$\leqq \left(\dfrac{b}{b+1-\cos x}\right)^n \sin x(\cos x)^{n-1}$

$\leqq \sin x(\cos x)^{n-1}$

↓ $0 \leqq x \leqq \dfrac{\pi}{2}$ で積分

$-\displaystyle\int_0^{\frac{\pi}{2}} \sin x(\cos x)^{n-1}dx$

$\leqq b^n a_n \leqq \displaystyle\int_0^{\frac{\pi}{2}} \sin x(\cos x)^{n-1}dx$

↓

$\left[\dfrac{1}{n}(\cos x)^n\right]_0^{\frac{\pi}{2}} \leqq b^n a_n \leqq -\left[\dfrac{1}{n}(\cos x)^n\right]_0^{\frac{\pi}{2}}$

$\therefore -\dfrac{1}{n} \leqq b^n a_n \leqq \dfrac{1}{n}$

$\displaystyle\lim_{n\to\infty}\left(-\dfrac{1}{n}\right) = \lim_{n\to\infty}\dfrac{1}{n} = 0$ より

$\displaystyle\lim_{n\to\infty} b^n a_n = 0$

a_1

$= \displaystyle\int_0^{\frac{\pi}{2}} \dfrac{\sin x}{b+1-b\cos x}dx$

$= \left[\dfrac{1}{b}\log|b+1-b\cos x|\right]_0^{\frac{\pi}{2}}$

$= \dfrac{1}{b}\log(b+1)$ //

(4)

a_{n+1}

$= \displaystyle\int_0^{\frac{\pi}{2}} \dfrac{\sin x(\cos x)^n}{(b+1-b\cos x)^{n+1}}dx$

$= \left[-\dfrac{1}{bn}\dfrac{(\cos x)^n}{(b+1-b\cos x)^n}\right]_0^{\frac{\pi}{2}}$

$-\dfrac{1}{b}\displaystyle\int_0^{\frac{\pi}{2}} \dfrac{\sin x(\cos x)^{n-1}}{(b+1-b\cos x)^n}dx$

$= -\dfrac{1}{b}a_n + \dfrac{1}{bn}$ //

(5) ↓

$a_n = -b a_{n+1} + \dfrac{1}{n}$

↓ $b = \dfrac{1}{2}$ とおく

$a_1 = -\dfrac{1}{2}a_2 + 1$

$= -\dfrac{1}{2}\left(-\dfrac{1}{2}a_3 + \dfrac{1}{2}\right) + 1$

$= \cdots$

$= \left(-\dfrac{1}{2}\right)^n a_{n+1} + \displaystyle\sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{k \cdot 2^{k-1}}$

↓

$\dfrac{1}{2}a_1 + \left(-\dfrac{1}{2}\right)^{n+1} a_{n+1} = \displaystyle\sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{k \cdot 2^k}$

(与式)

$= \displaystyle\lim_{n\to\infty}\left\{\dfrac{1}{2}a_1 + \left(-\dfrac{1}{2}\right)^{n+1} a_{n+1}\right\}$

$= \displaystyle\lim_{n\to\infty}\left(\log\dfrac{3}{2} + b^{n+1}a_{n+1}\right)$

$= \log\dfrac{3}{2}$ //

5

(1)

$|\alpha z+1|=2|z+\alpha|$

$|\alpha|=2$ のとき直線

$|\alpha z+1|^2=4|z+\alpha|^2$

$(\alpha z+1)(\overline{\alpha}\overline{z}+1)=4(z+\alpha)(\overline{z}+\overline{\alpha})$

$\vdots$

$(|\alpha|^2-4)z\overline{z}+(\alpha-4\overline{\alpha})z$

$+(\overline{\alpha}-4\alpha)\overline{z}+(1-4|\alpha|^2)=0$

$\Leftrightarrow z\overline{z}-\dfrac{\overline{\alpha}-4\alpha}{4-|\alpha|^2}z-\dfrac{\alpha-4\overline{\alpha}}{4-|\alpha|^2}\overline{z}=\dfrac{1-4|\alpha|^2}{4-|\alpha|^2}$

$\vdots$

$\left|z-\dfrac{\overline{\alpha}-4\alpha}{4-|\alpha|^2}\right|^2=\dfrac{4|\alpha^2-1|^2}{(4-|\alpha|^2)}$

中心 $\dfrac{\overline{\alpha}-4\alpha}{4-|\alpha|^2}$

$|\alpha|=2$ のとき

$\underset{s+ti}{(\alpha-4\overline{\alpha})}\underset{x+yi}{z}+\underset{s-ti}{(\overline{\alpha}-4\alpha)}\underset{x-yi}{\overline{z}}-15=0$

y を消すと

$x=s\cdot\dfrac{15}{2(s^2+t^2)}$

$y=-t\cdot\dfrac{15}{2(s^2+t^2)}$

絶対値が最小なのは

$z=s\cdot\dfrac{15}{2(s^2+t^2)}-i\cdot t\dfrac{15}{2(s^2+t^2)}$

$=\dfrac{15}{2(\alpha-4\overline{\alpha})}$

(2)

$(a,b,c)=(2,3,6)$