2022 横浜国大(理工)

1.

(1)

I

$= \left[\frac{1}{a}e^{at}\cos 2t\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{2}{a}e^{at}\sin 2t\,dt$

$= -\frac{1}{a}e^{\frac{a\pi}{2}} - \frac{1}{a}$

$+ \left[\frac{2}{a^2}e^{at}\sin 2t\right]_0^{\frac{\pi}{2}}$

$-\frac{4}{a^2}I$

$\Leftrightarrow \frac{a^2+4}{a^2}I = -\frac{1}{a}\left(e^{\frac{a\pi}{2}}+1\right)$

$\therefore I = -\frac{a}{a^2+4}\left(e^{\frac{a\pi}{2}}+1\right)$

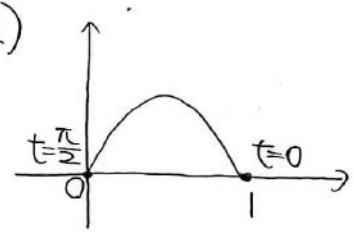
J

$= \left[\frac{1}{a}e^{at}\sin 2t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\frac{2}{a}e^{at}\cos 2t\,dt$

$= -\frac{2}{a}I$

$= \frac{2}{a^2+4}\left(e^{\frac{a\pi}{2}}+1\right)$

(2)



S

$= \int_0^1 y\,dx$

$= \int_{\frac{\pi}{2}}^0 y\frac{dx}{dt}\,dt$

$= \int_{\frac{\pi}{2}}^0 e^{-t}\sin t\left(-2e^{-2t}\cos t - e^{-2t}\sin t\right)dt$

$= \int_0^{\frac{\pi}{2}}\left(e^{-3t}\sin 2t + e^{-3t}\sin^2 t\right)dt$

$= \frac{2}{(-3)^2+4}\left(e^{-\frac{3\pi}{2}}+1\right)$

$+ \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(e^{-3t} - e^{-3t}\cos 2t\right)dt$

$= \frac{2}{13}\left(e^{-\frac{3\pi}{2}}+1\right)$

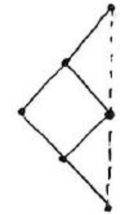
$-\frac{1}{6}\left[e^{-3t}\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left(\frac{3}{13}\right)\left(e^{-\frac{3\pi}{2}}+1\right)$

$= \frac{1}{26}\left(e^{-\frac{3\pi}{2}}+1\right) - \frac{1}{6}\left(e^{-\frac{3\pi}{2}}-1\right)$

$= \frac{-5e^{-\frac{3\pi}{2}}+8}{39}$

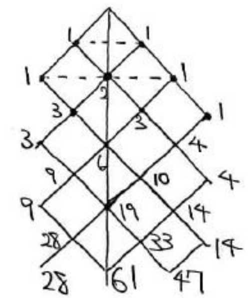
2.

(1)



図のような 2通り

(2)

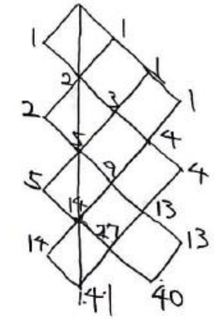


すべてたすと

$28+61+47=136$通り

(3)

$-1 \leqq m \leqq M \leqq 3$の場合は



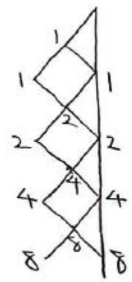
たして81通り

$m=-2$かつ$M \leqq 3$のときは

$136-81=55$通り

(4)

$m=-2$かつ$M=0$は



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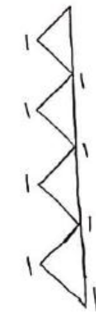


$16 \quad - \quad 1 = 15$

これを(3)から引くと

$55-15=40$

3.

(1)

$wz-w=2z-3$

$(w-2)z=w-3$

$z=\frac{w-3}{w-2} \quad (w \neq 2)$

$\downarrow \ |z| \geqq \frac{3}{2}$

$|w-3| \geqq \frac{3}{2}|w-2|$

$2|w-3| \geqq 3|w-2|$

2乗して

$$4|w-3|^2 \geqq 9|w-2|^2$$

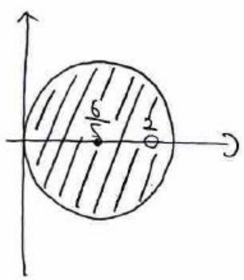
$\downarrow w = x+yi$

$$4\{(x-3)^2+y^2\} \geqq 9\{(x-2)^2+y^2\}$$

$$\Leftrightarrow 0 \geqq 5x^2 - 12x + 5y^2$$

$$\Leftrightarrow x^2 - \frac{12}{5}x + y^2 \leqq 0$$

$$\Leftrightarrow \left(x-\frac{6}{5}\right)^2 + y^2 \leqq \frac{36}{25}$$



(2)

$|z| \geqq r$ のとき

$$|w-3| \geqq r|w-2|$$

2乗して

$$|w-3|^2 \geqq r^2|w-2|^2$$

$\downarrow w = x+yi$

$$(x-3)^2+y^2 \geqq r^2\{(x-2)^2+y^2\}$$

$$\Leftrightarrow 0 \geqq (r^2-1)x^2 + (6-4r^2)x + (r^2-1)y^2 + 4r^2 - 9$$

$$\Leftrightarrow 0 \geqq x^2 + \frac{6-4r^2}{r^2-1}x + y^2 + \frac{4r^2-9}{r^2-1}$$

$$\Leftrightarrow \left(x+\frac{3-2r^2}{r^2-1}\right)^2 + y^2 \leqq \left(\frac{3-2r^2}{r^2-1}\right)^2 - \frac{4r^2-9}{r^2-1} = \left(\frac{r}{r^2-1}\right)^2$$

D_rの実軸部分は

$$\frac{2r^2-3}{r^2-1} - \frac{r}{r^2-1} \leqq x \leqq \frac{2r^2-3}{r^2-1} + \frac{r}{r^2-1}$$

ただし $x=2$ を除く.

ここで

$$\frac{2r^2-3}{r^2-1} - \frac{r}{r^2-1} = 2 - \frac{r+1}{r^2-1} = 2 - \frac{1}{r-1}$$

$$\frac{2r^2-3}{r^2-1} + \frac{r}{r^2-1} = 2 + \frac{1}{r+1}$$

つまり

$$2 - \frac{1}{r-1} \leqq x \leqq 2 + \underbrace{\frac{1}{r+1}}_{<3},\ x \neq 2$$

より求める条件は

$$-2 < 2 - \frac{1}{r-1} \leqq -1$$

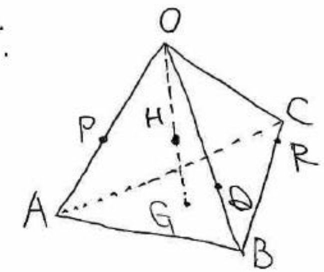
$$\Leftrightarrow -4 < -\frac{1}{r-1} \leqq -3$$

$$\Leftrightarrow 3 \leqq \frac{1}{r-1} < 4$$

$$\Leftrightarrow \frac{1}{3} \geqq r-1 > \frac{1}{4}$$

$$\therefore \frac{5}{4} < r \leqq \frac{4}{3}$$

4.



$\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c}$

とおく.

$$\overrightarrow{OP} = s\vec{a},\ \overrightarrow{OQ} = t\vec{b},$$

$$\overrightarrow{OR} = (1-u)\vec{b} + u\vec{c}$$

$$\overrightarrow{OH} = h\overrightarrow{OG} = \frac{h}{3}\vec{a} + \frac{h}{3}\vec{b} + \frac{h}{3}\vec{c}$$

また

$$\overrightarrow{OH} = \frac{1}{3}(\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}) = \frac{s}{3}\vec{a} + \frac{t}{3}\vec{b} + \frac{1-u}{3}\vec{b} + \frac{u}{3}\vec{c}$$

$\downarrow$

$$\begin{cases} s = h \\ t+1-u = h \\ u = h \end{cases} \rightarrow \quad t+1-s = s \quad \therefore s = \frac{t+1}{2}$$

同様に $u = \frac{t+1}{2}$

$0 < t < 1$ より $\frac{1}{2} < s < 1$

$|\overrightarrow{OP}|^2$

$$= |s\vec{a} - t\vec{b}|^2$$

$$= s^2 - 2st\frac{1}{2} + t^2$$

$$= \frac{t^2+2t+1}{4} - \frac{t^2+t}{2} + t^2$$

$$= \frac{3t^2+1}{4}$$

$|\overrightarrow{QR}|^2$

$$= \left|\frac{1-3t}{2}\vec{b} + \frac{t+1}{2}\vec{c}\right|^2$$

$$= \frac{(1-3t)^2 + (1-3t)(t+1) + (t+1)^2}{4}$$

$$= \frac{7t^2-6t+3}{4}$$

$\overrightarrow{QP} \cdot \overrightarrow{QR}$

$$= \left(\frac{t+1}{2}\vec{a} - t\vec{b}\right) \cdot \left(\frac{1-3t}{2}\vec{b} + \frac{t+1}{2}\vec{c}\right)$$

$$= \frac{(t+1)(1-3t)}{8} + \frac{(t+1)^2}{8} - \frac{t-3t^2}{2} - \frac{t^2+t}{4}$$

$$= \frac{4t^2-3t+1}{4}$$

(3)

S

$= \frac{1}{2}\sqrt{|\overrightarrow{OP}|^2|\overrightarrow{OR}|^2-(\overrightarrow{OP}\cdot\overrightarrow{OR})^2}$

$= \frac{1}{2}\sqrt{\frac{(3t^2+1)(7t^2-6t+3)}{16}-\frac{(4t^2-3t+1)^2}{16}}$

$= \frac{1}{8}\sqrt{5t^4+6t^3-t^2+2}$ //

$f(t)=5t^4+6t^3-t^2+2$ とおく

$f'(t)=20t^3+18t^2-2t$

$= 2t(10t-1)(t+1)$

t	$0 \cdots \frac{1}{10} \cdots 1$
$f'(t)$	$- \quad 0 \quad +$
$f(t)$	$\searrow \quad \nearrow$

$t=\frac{1}{10}$ のとき

$f(t)$とSが最小.

5.

$f(x)=2\log(\sin(ax))-(\log x)^2$

(1)

$f'(x)$

$= 2\cdot\frac{a\cos(ax)}{\sin(ax)}-2(\log x)\frac{1}{x}$

$= \frac{2a\cos(ax)}{\sin(ax)}-\frac{2\log x}{x}$ //

(2) $a=1$のとき

$f'(x)=\frac{2\cos x}{\sin x}-\frac{2\log x}{x}$

$= 2\left(\frac{1}{\tan x}-\frac{\log x}{x}\right)$

$g(x)=-\frac{\log x}{x}$ とおく

$g'(x)=-\frac{1-\log x}{x^2}<0$

$(1<x<\frac{\pi}{2})$

これより $g(x)$は単調減少.

$\frac{1}{\tan x}$も単調減少より

$f'(x)$も単調減少.

$\lim_{x\to 1+0} f'(x)=\frac{2}{\tan 1}>0$

$\lim_{x\to \frac{\pi}{2}-0} f'(x)=-\frac{4}{\pi}\log\frac{\pi}{2}<0$

より $f'(x)=0$ をみたす

解が $1<x<\frac{\pi}{2}$ に1つある.

この点で $f'(x)$の符号変化も

起こるので $f(x)$の極値を

与えるxは1つ.

(3)

$f'(x)=2\left(\frac{a}{\tan(ax)}-\frac{\log x}{x}\right)$

(2)と同様に $f'(x)$は単調減少

$\lim_{x\to\frac{\pi}{2}-0} f'(x)=\frac{2a}{\tan\frac{a\pi}{2}}-\frac{4}{\pi}\log\frac{\pi}{2}$

$> \frac{2a}{\tan\frac{a\pi}{2}}-\frac{2}{\pi}$ $\quad \frac{a\pi}{2}=\alpha$ とし

$= \frac{4}{\pi}\cdot\frac{\alpha}{\tan\alpha}-\frac{2}{\pi}$

$h(\alpha)=\frac{\alpha}{\tan\alpha}$ とおく $(\alpha\leqq\frac{\pi}{4})$

$h'(\alpha)=\frac{\tan\alpha-\frac{\alpha}{\cos^2\alpha}}{\tan^2\alpha}$

$= \frac{\sin\alpha\cos\alpha-\alpha}{\sin^2\alpha}$

$= \frac{\sin 2\alpha-2\alpha}{2\sin^2\alpha}$

$\leqq 0 \cdots ☆$

より

$\frac{4}{\pi}\cdot\frac{\alpha}{\tan\alpha}-\frac{2}{\pi}$

$> \frac{4}{\pi}\cdot\frac{\pi}{4}-\frac{2}{\pi}$

>0

これより $f'(x)>0$ $(1<x<\frac{\pi}{2})$

より $f(x)$は極値をもたない

- - - - - - - - - -

☆で

$i(x)=\sin x-x$ $(x\geqq 0)$

$i'(x)=\cos x-1\leqq 0$

より $i(x)$は単調減少

$i(0)=0$より $i(x)\leqq 0$

$\therefore \sin 2\alpha-2\alpha\leqq 0$