2022 日本医科大

[I]

$$W=\frac{(4+2i)Z+4-4i}{Z+2-2i}$$

$$WZ+(2-2i)W=(4+2i)Z+4-4i$$

$$(W-4-2i)Z=(-2+2i)W+4-4i$$

$$Z=\frac{(-2+2i)W+4-4i}{W-4-2i}$$

$\downarrow |Z|=2$

$$|(-2+2i)W+4-4i|=2|W-4-2i|$$

$$|(-1+i)W+2-2i|=|W-4-2i|$$

$$\underbrace{|-1+i|}_{\sqrt{2}}|W-2|=|W-4-2i|$$

$\downarrow$ 2乗

$$2(W-2)(\overline{W}-2)=(W-4-2i)(W-4+2i)$$

$$2W\overline{W}-4W-4\overline{W}+8$$

$$=W\overline{W}-(4-2i)W-(4+2i)\overline{W}+20$$

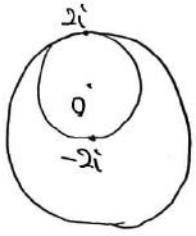
$$W\overline{W}-2iW+2i\overline{W}=12$$

$$(W+2i)(\overline{W}-2i)=16$$

$$|W+2i|^2=16$$

$$\therefore |W+2i|=4$$

$\alpha=-2i$ $r=4$



$Z=2i$のとき

$$W=\frac{4i-4+4}{2}=2i$$

より $\beta=2i$.

$$\frac{Z-W}{Z-\beta}$$

$$=\frac{Z-\frac{(4+2i)Z+4-4i}{Z+2-2i}}{Z-2i}$$

$$=\frac{Z(Z+2-2i)-(4+2i)Z-4+4i}{(Z-2i)(Z+2-2i)}$$

$$=\frac{Z(Z-2-4i)-4+4i}{(Z-2i)(Z+2-2i)}$$

$$=\frac{Z^2-(2+4i)Z-4+4i}{(Z-2i)(Z+2-2i)}$$

$$=\frac{(Z-2i)(Z-2-2i)}{(Z-2i)(Z+2-2i)}$$

$$=\frac{Z-2-2i}{Z+2-2i}$$

$$\sqrt{5}PQ\leqq BP$$

$$\Leftrightarrow \frac{PQ}{BP}\leqq\frac{1}{\sqrt{5}}$$

$$\downarrow$$

$$\left|\frac{Z-2-2i}{Z+2-2i}\right|\leqq\frac{1}{\sqrt{5}}$$

$$\downarrow$$

$$|Z-2-2i|^2\leqq\frac{1}{5}|Z+2-2i|^2$$

$$5|Z-2-2i|^2\leqq|Z+2-2i|^2$$

$\downarrow Z=x+yi$

$$5|x-2+(y-2)i|^2\leqq|x+2+(y-2)i|^2$$

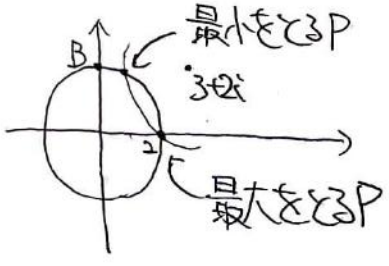
$$5\{(x-2)^2+(y-2)^2\}\leqq(x+2)^2+(y-2)^2$$

$$4x^2-24x+4y^2-16y+32\leqq0$$

$$x^2-6x+y^2-4y+8\leqq0$$

$$(x-3)^2+(y-2)^2\leqq5$$

$$\therefore |Z-3-2i|\leqq\sqrt{5}$$



$$\max BP=2\sqrt{2}$$

2つの交点を通る直線は

$$3(x-2)+2y=0$$

$$\Leftrightarrow y=-\frac{3}{2}x+3$$

$\downarrow x^2+y^2=4$

$$x^2+\left(-\frac{3}{2}x+3\right)^2=4$$

$$\Leftrightarrow 4x^2+(-3x+6)^2=16$$

$$\Leftrightarrow 13x^2-36x+20=0$$

$$\Leftrightarrow (13x-10)(x-2)=0$$

$$x=\frac{10}{13},2$$

$$y=\frac{24}{13},0$$

$$\min BP=\sqrt{\left(\frac{10}{13}\right)^2+\left(\frac{2}{13}\right)^2}$$

$$=\frac{\sqrt{104}}{13}$$

$$=\frac{2\sqrt{26}}{13}$$

[II]

問1.

P

$= P_{赤}(A)$

$= \dfrac{\frac{1}{6}\cdot\frac{x}{n}}{\frac{1}{6}\cdot\frac{x}{n}+\frac{5}{6}\cdot\frac{y}{n}} = \dfrac{x}{x+5y}$ //

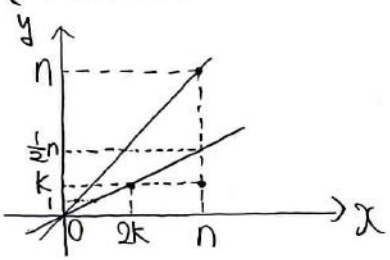
問2.

$\dfrac{1}{6} \leqq \dfrac{x}{x+5y} \leqq \dfrac{2}{7}$

$y \leqq x$ $\qquad$ $y \geqq \frac{1}{2}x$

つまり

$$\begin{cases} \frac{1}{2}x \leqq y \leqq x \\ 1 \leqq x \leqq n \\ 1 \leqq y \leqq n \end{cases}$$



$N(n)$

$= (1+n)\times n\times\frac{1}{2}$

$-\displaystyle\sum_{k=1}^{[\frac{n}{2}]}(n-2k)$

$= \frac{n}{2}(n+1) - n\cdot\left[\frac{n}{2}\right] + 2\cdot\frac{1}{2}\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)$

$= \frac{n}{2}(n+1) + \left[\frac{n}{2}\right]\left(-n+\left[\frac{n}{2}\right]+1\right)$

$= \begin{cases} \frac{n}{2}(n+1)+\frac{n}{2}\left(-n+\frac{n}{2}+1\right) & (n\text{が偶数}) \\ \frac{n}{2}(n+1)+\frac{n-1}{2}\left(-n+\frac{n-1}{2}+1\right) & (n\text{が奇数}) \end{cases}$

$= \begin{cases} \frac{1}{4}n^2+n & (n\text{が偶数}) \\ \frac{1}{4}n^2+n-\frac{1}{4} & (n\text{が奇数}) \end{cases}$ //

問3.

nが偶数のとき

$\frac{1}{4}n^2+n < 2022$

$\Leftrightarrow n(n+4) < 8088$

これをみたす最大のnは86

nが奇数のとき

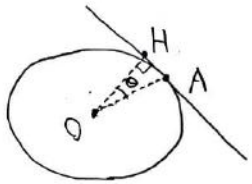
$\frac{1}{4}n^2+n-\frac{1}{4} < 2022$

$\Leftrightarrow n(n+4) < 8089$

これをみたす最大のnは87

以上より求めるnは87 //

[III]



問1

連立して

$b^2x^2+a^2(-mx+k)^2=a^2b^2$

$\Leftrightarrow (b^2+a^2m^2)x^2-2a^2mkx+a^2k^2-a^2b^2=0$

$\frac{D}{4} = (-a^2mk)^2-(b^2+a^2m^2)(a^2k^2-a^2b^2) = 0$

$\Leftrightarrow a^2m^2k^2-(b^2+a^2m^2)(k^2-b^2)=0$

$\Leftrightarrow -b^2k^2+b^4+a^2b^2m^2=0$

$\therefore k^2 = a^2m^2+b^2$

このとき

$x = \dfrac{2a^2mk}{2(b^2+a^2m^2)}$

$= \dfrac{a^2m}{\sqrt{a^2m^2+b^2}}$

$y = -\dfrac{a^2m^2}{\sqrt{a^2m^2+b^2}} + \sqrt{a^2m^2+b^2}$

$= \dfrac{b^2}{\sqrt{a^2m^2+b^2}}$

$A\left(\dfrac{a^2m}{\sqrt{a^2m^2+b^2}}, \dfrac{b^2}{\sqrt{a^2m^2+b^2}}\right)$ //

問2. Oと$mx+y-k=0$の距離

$h = \dfrac{|-k|}{\sqrt{m^2+1}} = \sqrt{\dfrac{a^2m^2+b^2}{m^2+1}}$ //

問3.

$OA = \dfrac{\sqrt{a^4m^2+b^4}}{\sqrt{a^2m^2+b^2}}$

$\cos\theta = \dfrac{\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}} \times \dfrac{\sqrt{a^2m^2+b^2}}{\sqrt{a^4m^2+b^4}}$

$= \dfrac{a^2m^2+b^2}{\sqrt{m^2+1}\sqrt{a^4m^2+b^4}}$

$\sin\theta$

$= \sqrt{1-\dfrac{a^4m^4+2a^2b^2m^2+b^4}{(m^2+1)(a^4m^2+b^4)}}$

$= \dfrac{\sqrt{m^2(a^2-b^2)^2}}{\sqrt{m^2+1}\sqrt{a^4m^2+b^4}}$

$= \dfrac{m(a^2-b^2)}{\sqrt{(m^2+1)(a^4m^2+b^4)}}$ //

解4

$\sin\theta$

$$= \frac{a^2-b^2}{\sqrt{\left(m+\frac{1}{m}\right)\left(a^4m+\frac{b^4}{m}\right)}}$$

$$= \frac{a^2-b^2}{\sqrt{a^4m^2+\frac{b^4}{m^2}+a^4+b^4}}$$

$a^4m^2>0,\ \frac{b^4}{m^2}>0$

(相加平均)≧(相乗平均)より

$\sin\theta$

$$\leqq \frac{a^2-b^2}{\sqrt{2a^4m^2\cdot\frac{b^4}{m^2}+a^4+b^4}}$$

$$= \frac{a^2-b^2}{a^2+b^2}$$

等号成立は $a^4m^2=\frac{b^4}{m^2}$

つまり $m=\frac{b}{a}$ のとき.

このとき $M(a,b)=\frac{a^2-b^2}{a^2+b^2}$ //

解5.

$\frac{b}{a}=x$ とおくと

$$M(a,b)=\frac{1-x^2}{1+x^2}=\frac{2}{1+x^2}-1$$

$$(a-b)^2+(b-1)^2\leqq\frac{3}{4}$$

$$\Longleftrightarrow (a-ax)^2+(ax-1)^2-\frac{3}{4}\leqq 0$$

$$\Longleftrightarrow \underbrace{(2x^2-2x+1)}_{>0}a^2-2xa+\frac{1}{4}\leqq 0$$

これをみたす a が存在するので条件は

$$\begin{cases} 0<x<1 \\ \frac{D}{4}=x^2-\frac{1}{4}(2x^2-2x+1) \\ \quad =\frac{1}{2}x^2+\frac{x}{2}-\frac{1}{4}\geqq 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 0<x<1 \\ 2x^2+2x-1\geqq 0 \end{cases}$$

解くと $\frac{-1+\sqrt{3}}{2}\leqq x<1$

$x=\frac{-1+\sqrt{3}}{2}$ のとき $M(a,b)$ が最大をとるので

$$\max M(a,b)=\frac{8}{4+(-1+\sqrt{3})^2}-1$$

$$=\frac{3+4\sqrt{3}}{13}$$ //

[IV] 解1

$$f(x)=\left(\frac{1-x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

$$f'(x)=\frac{1}{2}\left(\frac{1-x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}}\right)^{-\frac{1}{2}}$$

$$\times\frac{-\frac{1}{2}x^{-\frac{1}{2}}(1+x^{\frac{1}{2}})-(1-x^{\frac{1}{2}})\frac{1}{2}x^{-\frac{1}{2}}}{(1+x^{\frac{1}{2}})^2}$$

$$=\cdots$$

$$=\frac{f(x)}{2\sqrt{x}(x-1)}$$

$f''(x)$

$$=\frac{f'(x)2\sqrt{x}(x-1)-f(x)2\left\{\frac{3}{2}\sqrt{x}-\frac{1}{2\sqrt{x}}\right\}}{4x(x-1)^2}$$

$$=\cdots$$

$$=\frac{-3f(x)(3x-\sqrt{x}-1)}{4x\sqrt{x}(x-1)^2}$$

$$=-\frac{3f(x)}{4x\sqrt{x}(x-1)^2}\left(\sqrt{x}-\frac{1-\sqrt{13}}{6}\right)\left(\sqrt{x}-\frac{1+\sqrt{13}}{6}\right)$$

$0<x<1$ をふまえると

$$\sqrt{x}=\frac{1+\sqrt{13}}{6}$$

$$x=\left(\frac{1+\sqrt{13}}{6}\right)^2=\frac{7+\sqrt{13}}{18}$$

のとき $f''(x)=0$.

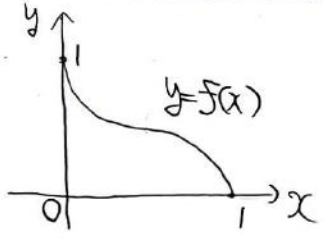
$$f\left(\frac{7+\sqrt{13}}{18}\right)=\sqrt{\frac{1-\frac{1+\sqrt{13}}{6}}{1+\frac{1+\sqrt{13}}{6}}}$$

$$=\cdots$$

$$=\frac{\sqrt{12-2\sqrt{13}}}{3}$$

x	$0\cdots$	$\frac{7+\sqrt{13}}{18}$	$\cdots 1$
$f(x)$	$-$	$-$	$-$
$f''(x)$	$+$	0	$-$
$f(x)$	⤷		⤵

変曲点 $\left(\frac{7+\sqrt{13}}{18},\ \frac{\sqrt{12-2\sqrt{13}}}{3}\right)$ //



$y=\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$ を x について解くと

$$x=\left(\frac{1-y^2}{1+y^2}\right)^2$$

$$V=\pi\int_0^1\left(\frac{1-y^2}{1+y^2}\right)^4dy$$

$y=\tan\theta$ とおくと

$$V=\pi\int_0^{\frac{\pi}{4}}\left(16\cos^6\theta-32\cos^4\theta+24\cos^2\theta-8+\frac{1}{\cos^2\theta}\right)d\theta$$

頑張って解くと $V=\pi\left(\frac{8}{3}-\frac{3}{4}\pi\right)$ //