2022 杏林大(医)

I

(1) $\cos 2\theta = \underline{-2}\sin^2\theta + \underline{1}$

$\sin 3\theta = \underline{-4}\sin^3\theta + \underline{3}\sin\theta$

(2)

y

$= -\frac{1}{12}(-4\sin^3\theta + 3\sin\theta)$

$\quad + \frac{3}{8}(-2\sin^2\theta + 1) - \frac{3}{4}\sin\theta$

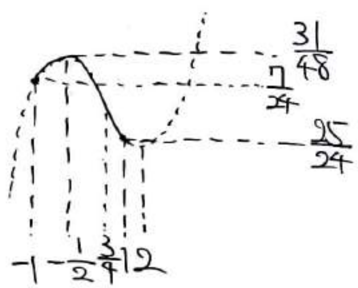
$= \frac{1}{3}\sin^3\theta - \frac{3}{4}\sin^2\theta - \sin\theta + \frac{3}{8}$

$= \frac{1}{3}t^3 - \frac{3}{4}t^2 - t + \frac{3}{8}$

$\frac{dy}{dt} = t^2 - \frac{3}{2}t - 1$

$= \frac{1}{2}(2t^2 - 3t - 2)$

$= \frac{1}{2}(2t+1)(t-2)$



$t=1$ つまり $\theta = \frac{1}{2}\pi$ で最小値 $\underline{-\frac{25}{24}}$

$t = \underline{-\frac{1}{2}}$ のとき最大値 $\underline{\frac{31}{48}}$

$\frac{dy}{d\theta} = \frac{dy}{dt}\cdot\frac{dt}{d\theta}$

$= \frac{1}{2}(2\sin\theta+1)(\sin\theta-2)\cos\theta$

$\frac{dy}{d\theta}$ の符号変化がおこる θ は

$0 \leqq \theta < 2\pi$ で $\underline{4}$ コ.

II

(1)

$\int x e^{-3x} dx$

$= -\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x} + C$

$= -\underline{\frac{3x+1}{9}}e^{-3x} + C$

$\int x^2 e^{-3x} dx$

$= -\frac{x^2}{3}e^{-3x} - \frac{2x}{9}e^{-3x} - \frac{2}{27}e^{-3x} + C$

$\boxed{x^2 e^{-3x} - \frac{2x}{3}e^{-3x}}\ \boxed{+\frac{2x}{3}e^{-3x} - \frac{2}{9}e^{-3x}}$

$= -\underline{\frac{9x^2+6x+2}{27}}e^{-3x} + C$

$\int_0^1 |(9x^2-1)e^{-3x}| dx$

$= \int_0^{\frac{1}{3}} \underbrace{(1-9x^2)e^{-3x}}_{-f(x)} dx + \int_{\frac{1}{3}}^1 \underbrace{(9x^2-1)e^{-3x}}_{f(x)} dx$

$= \left[-F(x)\right]_0^{\frac{1}{3}} + \left[F(x)\right]_{\frac{1}{3}}^1$

$= F(0) + F(1) - 2\cdot F\left(\frac{1}{3}\right)$

$\int f(x) dx$

$= 9\int x^2 e^{-3x} dx - \int e^{-3x} dx$

$= -\frac{9x^2+6x+2}{3}e^{-3x} + \frac{1}{3}e^{-3x} + C$

$= -\frac{9x^2+6x+1}{3}e^{-3x} + C$

$= -\frac{(3x+1)^2}{3}e^{-3x} + C$

$= -\frac{1}{3} - \frac{16}{3}e^{-3} + \frac{8}{3}e^{-1}$

$= \underline{\frac{1}{3}(-1 + 8e^{-1} - 16e^{-3})}$

(2)

$f(x) = (px^2 + qx + r)e^{-3x}$

$f'(x) = (2px + q)e^{-3x}$

$\quad - (3px^2 + 3qx + 3r)e^{-3x}$

$= (-3px^2 + (2p-3q)x + q - 3r)e^{-3x}$

$f'(0) = q - 3r = 0$ $\quad q = 3r$

$f'(1) = -3p + 2p - 3q = 0$

$\Leftrightarrow p = -9r$

$f'(x) = (27rx^2 - 27rx)e^{-3x}$

$= 27rx(x-1)e^{-3x}$

$r > 0$ だと $x = 0$ で極大.

$\underline{p = -9r}$, $\underline{q = 3r}$, $r > 0$ $\underline{①}$

$f(1) = (-9r + 3r + r)e^{-3}$

$= -5re^{-3} = -1$

$\Leftrightarrow r = \frac{1}{5}e^3$

$f(0) = r = \underline{\frac{e^3}{5}}$ ← 極大値

$\int_0^1 f(x) dx$

$= \int_0^1 (-9rx^2 + 3rx + r)e^{-3x} dx$

$= -r\int_0^1 (9x^2 - 3x - 1)e^{-3x} dx$

$= -r\left[-\frac{9x^2+6x+2}{3}e^{-3x} + \frac{3x+1}{3}e^{-3x} + \frac{e^{-3x}}{3}\right]$

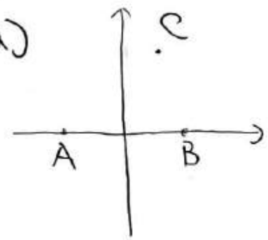
$= -r\left[(-3x^2 - x)e^{-3x}\right]_0^1$

$= r\cdot 4e^{-3} = \underline{\frac{4}{5}}$

III

(1)

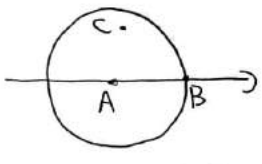
(a)



Cは第1象限

(b) $\angle ABC = \angle ACB$ なら

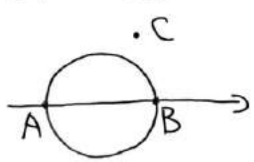
$AB = AC$



$\angle ABC < \angle ACB$ なのは上図.

Cが ①の ①に存在.

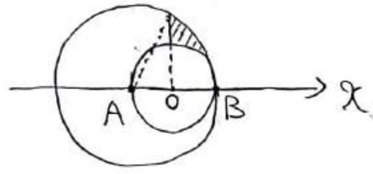
(c) $\angle ACB = \frac{\pi}{2}$ ならCは下の円周上.



$\angle ACB < \frac{\pi}{2}$ なのは上図.

Cは③の③に存在

(d) (a), (b), (c)を答える.



斜線部分は

$4\pi \cdot \frac{1}{6} - 1 \cdot \sqrt{3} \cdot \frac{1}{2} - \pi \cdot \frac{1}{4}$

$= \frac{5}{12}\pi - \frac{\sqrt{3}}{2}$

(2)

(a) $\angle ADC = \frac{\pi}{2}$ のとき

Dは ACを直径とした球面上 ②

(b)

$\angle ADC = \frac{\pi}{2}$ より $D(X, Y, Z)$ として

$\left(X - \frac{S-1}{2}\right)^2 + \left(Y - \frac{t}{2}\right)^2 + Z^2 = \left(\frac{S+1}{2}\right)^2 + \frac{t^2}{4}$

$\angle BDC = \frac{\pi}{2}$ より同様に

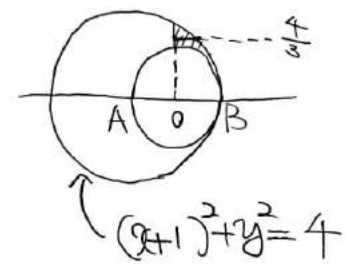
$\left(X - \frac{S+1}{2}\right)^2 + \left(Y - \frac{t}{2}\right)^2 + Z^2 = \left(\frac{S-1}{2}\right)^2 + \frac{t^2}{4}$

$X = S$ より

$\left(Y - \frac{t}{2}\right)^2 + Z^2 = \frac{t^2}{4}$

Dは $\left(S, \frac{t}{2}, 0\right)$ を中心, 半径 $\frac{t}{2}$ の円 ⑥ ⑥

(c)



$(x+1)^2 + y^2 = 4$

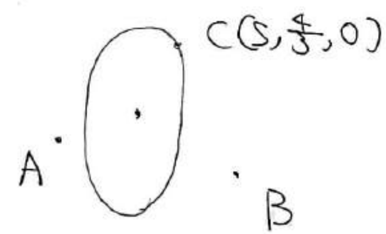
$y = \frac{4}{3}$ のとき

$(x+1)^2 = 4 - \frac{16}{9} = \frac{20}{9}$

$\therefore x + 1 = \frac{2\sqrt{5}}{3} \quad (x+1>0)$

$x = -1 + \frac{2\sqrt{5}}{3}$

よって $0 < S < -1 + \frac{2\sqrt{5}}{3}$



Dは ABを直径とする球面上

$X^2 + Y^2 + Z^2 = 1 \quad \cdots ①$

また

$X = S$ かつ $\left(Y - \frac{2}{3}\right)^2 + Z^2 = \frac{4}{9} \quad \cdots ②$

①, ②より

$Y^2 + Z^2 = 1 - S^2$

$-) \ \left(Y - \frac{2}{3}\right)^2 + Z^2 = \frac{4}{9}$

$\frac{4}{3}Y = 1 - S^2$

$\iff Y = \frac{3}{4}(1 - S^2)$

$Z^2 = 1 - S^2 - \frac{9}{16}(1 - S^2)^2$

ちなみに

$\triangle ABC = 2 \cdot \frac{4}{3} \cdot \frac{1}{2} = \frac{4}{3}$

より Zが最大のとき

四面体ABCDが最大.

$1 - S^2 = u$ とおくと

$Z^2 = -\frac{9}{16}u^2 + u$

$= -\frac{9}{16}\left(u^2 - \frac{16}{9}u\right)$

$= -\frac{9}{16}\left(u - \frac{8}{9}\right)^2 + \frac{9}{16} \cdot \frac{64}{81}$

$\underset{1-S^2}{u} = \frac{8}{9}$ つまり $S = \frac{1}{3}$ のとき

$\max Z = \sqrt{\frac{9}{16} \cdot \frac{64}{81}} = \frac{2}{3}$

つまり Vは $S = \frac{1}{3}$ のとき

最大値 $\frac{4}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{8}{27}$