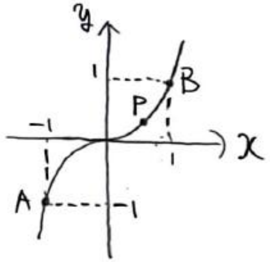
2021 早稲田大(理工系)

[I]

(1)

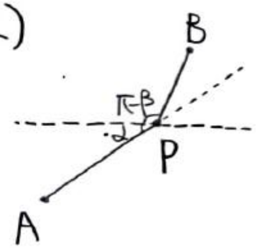


$\tan\alpha = \dfrac{t^3+1}{t+1} = \underline{t^2-t+1}$ //

$\tan\beta = \dfrac{1-t^3}{1-t} = 1+t+t^2$

$= \underline{t^2+t+1}$ //

(2)



$\tan\angle APB$

$= \tan(\pi-\beta+\alpha)$

$= -\tan(\beta-\alpha)$

$= -\dfrac{\tan\beta-\tan\alpha}{1+\tan\beta\tan\alpha}$

$= \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$= \dfrac{-2t}{1+(t^2+1-t)(t^2+1+t)}$

$= \dfrac{-2t}{1+(t^2+1)^2-t^2}$

$= \underline{\dfrac{-2t}{t^4+t^2+2}}$ //

(3)

$f(t) = \dfrac{-2t}{t^4+t^2+2}$ とおく.

$f'(t) = \dfrac{-2(t^4+t^2+2)+2t(4t^3+2t)}{(t^4+t^2+2)^2}$

$= \dfrac{6t^4+2t^2-4}{(t^4+t^2+2)^2}$

$= \dfrac{2(3t^2-2)(t^2+1)}{(t^4+t^2+2)^2}$

t	0	$\cdots$	$\sqrt{\frac{2}{3}}$	$\cdots$	1
$f'(t)$		$-$	0	$+$	
$f(t)$		$\searrow$		$\nearrow$	

$t = \sqrt{\frac{2}{3}} = \underline{\frac{\sqrt{6}}{3}}$ のとき最小. //

[II]

(1) $x^2 = t$ として

t^3 を t^2-t+1 で割った余りを考える.

$$\begin{array}{r}
1 \quad 1 \qquad\qquad \\
1\ -1\ \ 1\ )\ 1 \qquad\qquad\quad \\
1\ -1\ \ 1 \qquad \\
1\ -1 \qquad \\
1\ -1\ \ 1 \\
-1
\end{array}$$

$t^3 = (t^2-t+1)(t+1)-1$

$\therefore x^6 = (x^4-x^2+1)(x^2+1)-1$

余りは $\underline{-1}$ //

(2)

$x^{2020}+1 = x^{2020}+x^{2018} \qquad +x^4$
$\qquad -x^{2018}-x^{2016}+\cdots+-x^2$
$\qquad +x^{2016}+x^{2014} \qquad +1$

$= (x^4-x^2+1)(x^{2016}+x^{2014}+\cdots+x^2+1)$

$\Downarrow$ 両辺 $\times x$

$x^{2021}+x = (x^4-x^2+1)(x^{2017}+x^{2015}+\cdots+x^3+x)$

$\Leftrightarrow x^{2021} = f(x)(x^{2017}+x^{2015}+\cdots+x^3+x)-x$

余りは $\underline{-x}$ //

(3)

$(x^2-1)^{3m}-1$ (m は自然数) が

$f(x)$ で割り切れることを m に関する

数学的帰納法で示す.

(i) $m=1$ のとき

$(x^2-1)^3-1$

$= x^6-3x^4+3x^2-2$

$= (x^4-x^2+1)(x^2-2)$

$= f(x)(x^2-2)$ $f(x)$ で割り切れる.

(ii) $m=k$ のとき $f(x)$ で割り切れる

とすると

$(x^2-1)^{3k}-1 = f(x)Q(x)$

と仮定する.

$m=k+1$ のとき

$(x^2-1)^{3k+3}$

$= (x^2-1)^{3k}(x^2-1)^3$

$= \{f(x)Q(x)+1\}(x^2-1)^3$

$= f(x)Q(x)(x^2-1)^3+(x^2-1)^3$

$(x^2-1)^{3(k+1)}-1$

$= f(x)Q(x)(x^2-1)^3+(x^2-1)^3-1$

$= f(x)\{Q(x)(x^2-1)^3+x^2-2\}$

よってこのときも $f(x)$ で割り切れる.

(i)(ii) よりすべての自然数 m に

ついて $f(x)$ で割り切れる.

[III]

(1) $\alpha^2=3+4i$

$\beta^2=-\frac{3}{4}-i$

xy平面で考えると

$C(3,4)$, $D(-\frac{3}{4},-1)$

$\overrightarrow{OD}=-\frac{1}{4}\overrightarrow{OC}$

より, O, C, Dは一直線上.

(2) $A(2,1)$, $B(-\frac{1}{2},1)$

直線AB: $y=1$

$z=t+i$ $(t\in\mathbb{R})$とおくと

$z^2=t^2-1+2ti$

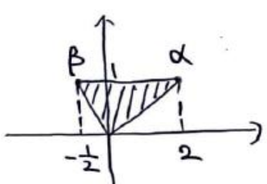
$\hookrightarrow \begin{cases} x=t^2-1 \\ y=2t \end{cases}$

$\Downarrow$

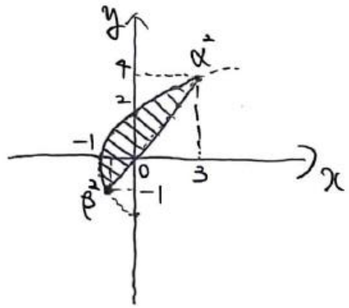
$x=\left(\frac{y}{2}\right)^2-1$

$\therefore \underline{y^2=4(x+1)}$ //

(3)



求める領域は線分OC, OD, そして $y^2=4(x+1)$ で囲まれた領域



の斜線部分. 境界を含む.

(4) 放物線: $x=\frac{y^2}{4}-1$

直線CD: $x=\frac{3}{4}y$

(Kの面積)

$=\int_{-1}^{4}\left\{\frac{3}{4}y-\left(\frac{y^2}{4}-1\right)\right\}dy$

$=-\frac{1}{4}\int_{-1}^{4}(y+1)(y-4)dy$

$=-\frac{1}{4}\left\{-\frac{1}{6}(4-(-1))^3\right\}$

$=\frac{1}{24}\cdot 5^3=\underline{\frac{125}{24}}$ //

[IV]

(1) $n=2$, $k=3$のとき

$P_0=\underline{0}$ //

$P_3=\frac{2}{2^3}=\underline{\frac{1}{4}}$ //

$P_2=\underline{0}$ //

$P_1=1-\frac{1}{4}=\underline{\frac{3}{4}}$ //

(2) $n>2$, $k=2$のとき

$P_2=\frac{n}{n^2}=\underline{\frac{1}{n}}$ //

$P_1=\frac{n(n-1)}{n^2}=\underline{\frac{n-1}{n}}$ //

$P_0=\underline{0}$ //

$n=2$, $k=2$のとき

$P_2=\frac{2}{2^2}=\underline{\frac{1}{2}}$ //

$P_1=\underline{0}$ // $P_0=\underline{\frac{1}{2}}$ //

(3) $n>3$, $k=3$のとき

$P_3=\frac{n}{n^3}=\underline{\frac{1}{n^2}}$ //

$P_2=\frac{{}_3C_2\cdot n(n-1)}{n^3}=\underline{\frac{3(n-1)}{n^2}}$ //

$P_0=\underline{0}$ //

$P_1=1-\frac{1}{n^2}-\frac{3(n-1)}{n^2}$

$=\frac{n^2-1-3n+3}{n^2}$

$=\underline{\frac{n^2-3n+2}{n^2}}$ //

$n=3$, $k=3$のとき

$P_3=\frac{3}{3^3}=\underline{\frac{1}{9}}$ //

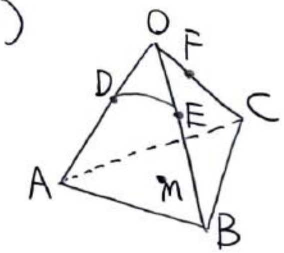
$P_1=\underline{0}$ //

$P_0=\frac{3\cdot2\cdot1}{3^3}=\underline{\frac{2}{9}}$ //

$P_2=1-\frac{1}{9}-\frac{2}{9}=\underline{\frac{2}{3}}$ //

[V] 正三角形の外心と重心は同じ.

(1)



$\overrightarrow{OM} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$

$\overrightarrow{OD} = d\vec{a}$ とおくと

$\overrightarrow{DM} = \left(\frac{1}{3}-d\right)\vec{a} + \frac{1}{3}\vec{b} + \frac{1}{3}\vec{c}$

$|\overrightarrow{DM}|^2$

$= \left(\frac{1}{3}-d\right)^2|\vec{a}|^2 + \frac{1}{9}|\vec{b}|^2 + \frac{1}{9}|\vec{c}|^2$

$\quad + \frac{2}{3}\left(\frac{1}{3}-d\right)\vec{a}\cdot\vec{b} + \frac{2}{9}\vec{b}\cdot\vec{c}$

$\quad + \frac{2}{3}\left(\frac{1}{3}-d\right)\vec{c}\cdot\vec{a}$

$= \left(\frac{1}{3}-d\right)^2|\vec{a}|^2 + \frac{2}{9}|\vec{a}|^2$

$\frac{1}{3}\left(\frac{1}{3}-d\right)|\vec{a}|^2 \times 2 + \frac{1}{9}|\vec{a}|^2$

$= \left\{\left(\frac{1}{3}-d\right)^2 + \frac{1}{3} + \frac{2}{3}\left(\frac{1}{3}-d\right)\right\}|\vec{a}|^2$

$= \left(d^2 - \frac{4}{3}d + \frac{2}{3}\right)|\vec{a}|^2 = \left\{\left(\frac{\sqrt{3}}{3}\right)|\vec{a}|\right\}^2$

$= d^2 - \frac{4}{3}d + \frac{1}{3} = 0$

$d = \frac{1}{3} \quad (0<d<1)$

同様に考え

$\overrightarrow{OD} = \frac{1}{3}\vec{a}, \overrightarrow{OE} = \frac{1}{3}\vec{b}, \overrightarrow{OF} = \frac{1}{3}\vec{c}$ //

$AM = \frac{\sqrt{3}}{3}|\vec{a}|$

$\overrightarrow{OG} = g\frac{\vec{a}+\vec{b}}{2} = \frac{g}{2}(\vec{a}+\vec{b})$

とおくと

$\overrightarrow{DG} = \left(\frac{g}{2}-\frac{1}{3}\right)\vec{a} + \frac{g}{2}\vec{b}$

$|\overrightarrow{DG}|^2$

$= \left(\frac{g}{2}-\frac{1}{3}\right)^2|\vec{a}|^2 + g\left(\frac{g}{2}-\frac{1}{3}\right)\vec{a}\cdot\vec{b}$

$\quad + \frac{g^2}{4}|\vec{b}|^2$

$= \left\{\left(\frac{g}{2}-\frac{1}{3}\right)^2 + \frac{g}{2}\left(\frac{g}{2}-\frac{1}{3}\right) + \frac{g^2}{4}\right\}|\vec{a}|^2$

$= \left(\frac{3}{4}g^2 - \frac{1}{2}g + \frac{1}{9}\right)|\vec{a}|^2$

$\overrightarrow{AG} = \left(\frac{g}{2}-1\right)\vec{a} + \frac{g}{2}\vec{b}$

$|\overrightarrow{AG}|^2$

$= \left\{\left(\frac{g}{2}-1\right)^2 + \frac{g}{2}\left(\frac{g}{2}-1\right) + \frac{g^2}{4}\right\}|\vec{a}|^2$

$= \left(\frac{3}{4}g^2 - \frac{3}{2}g + 1\right)|\vec{a}|^2$

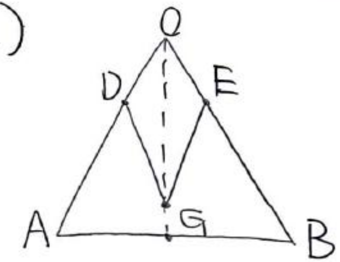
$|\overrightarrow{DG}|^2 = |\overrightarrow{AG}|^2$ より

$\frac{3}{4}g^2 - \frac{1}{2}g + \frac{1}{9} = \frac{3}{4}g^2 - \frac{3}{2}g + 1$

$\Leftrightarrow g = \frac{8}{9}$

$\therefore \overrightarrow{OG} = \frac{4}{9}(\vec{a}+\vec{b})$ //

(2)



$S_1 : S_2$

$= 1 : \frac{1}{3} \times \frac{8}{9}$

$= 27 : 8$ //