2021 筑波大

[1]〜[3] 数IIB

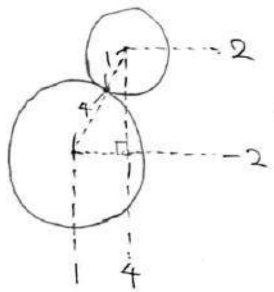
[4]〜[6] 数III

[1]

$C_1: (x-1)^2+(y+2)^2=16$

$C_2: (x-4)^2+(y-2)^2=20-k$

(1)



半径の和が5なので

$4+\sqrt{20-k}=5$

$\sqrt{20-k}=1$

$\therefore \underline{k=19}$ //

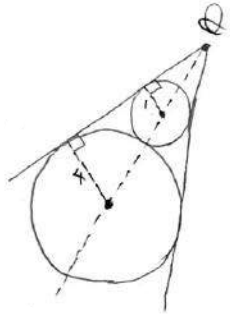
(2) $P(x_P, y_P)$ とおくと

$x_P = 1+4\times\frac{3}{5}=\frac{17}{5}$

$y_P=-2+4\times\frac{4}{5}=\frac{6}{5}$

$\therefore \underline{P\left(\frac{17}{5}, \frac{6}{5}\right)}$ //

(3)



Qは $(4,2)$, $(1,-2)$ を $1:4$ に外分した点より

$\left(\frac{-4\cdot4+1\cdot1}{1-4}, \frac{-4\cdot2+1(-2)}{1-4}\right)$

$\therefore \underline{Q\left(5, \frac{10}{3}\right)}$ //

[2]

(1)

$t=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$

$\left(-\frac{\pi}{4}<\theta+\frac{\pi}{4}<\frac{3}{4}\pi\right)$

$\underline{-1<t\leqq\sqrt{2}}$ //

(2)

$\sin^3\theta+\cos^3\theta$

$=(\sin\theta+\cos\theta)^3-3\sin\theta\cos\theta(\sin\theta+\cos\theta)$

$=t^3-3\frac{(\sin\theta+\cos\theta)^2-1}{2}t$

$=t^3-\frac{3}{2}(t^3-t)$

$=\underline{-\frac{1}{2}t^3+\frac{3}{2}t}$ //

$\cos4\theta$

$=1-2\sin^2 2\theta$

$=1-8\sin^2\theta\cos^2\theta$

$=1-8\left(\frac{t^2-1}{2}\right)^2$

$=1-2(t^2-1)^2$

$=\underline{-2t^4+4t^2-1}$ //

(3)

$\sin^3\theta+\cos^3\theta=\cos4\theta$

$-\frac{1}{2}t^3+\frac{3}{2}t=-2t^4+4t^2-1$

$2t^4-\frac{1}{2}t^3-4t^2+\frac{3}{2}t+1=0$

$$\begin{array}{ccccc|c} 2 & -\frac{1}{2} & -4 & \frac{3}{2} & +1 & \\ & 2 & \frac{3}{2} & -\frac{5}{2} & -1 & 1 \\ \hline 2 & \frac{3}{2} & -\frac{5}{2} & -1 & & \\ & 2 & \frac{7}{2} & 1 & & 1 \\ \hline 2 & \frac{7}{2} & 1 & & & \end{array}$$

$(t-1)^2\left(2t^2+\frac{7}{2}t+1\right)=0$

$\therefore t=1, \frac{-7\pm\sqrt{49-32}}{8}$

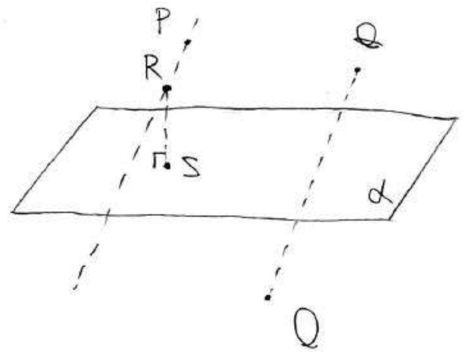
$-1<t\leqq\sqrt{2}$ にあるのは

$\underline{t=1, \frac{-7+\sqrt{17}}{8}}$ //

[3]

$\alpha: \frac{x}{-2}+y+z=1$

$\Leftrightarrow x-2y-2z+2=0$



$\overrightarrow{OR}=\overrightarrow{OP}+k\overrightarrow{OQ}$

$=\begin{pmatrix} k \\ 5+k \\ 5+k \end{pmatrix}$

直線RS: $\begin{cases} x=k+t \\ y=5+k-2t \\ z=5+k-2t \end{cases}$ $(t\in\mathbb{R})$

これとαを連立すると

$k+t-2(s+k-2t)-2(s+k-2t)+2=0$

$\iff 9t=3k+18$

$\therefore t=\frac{1}{3}k+2$

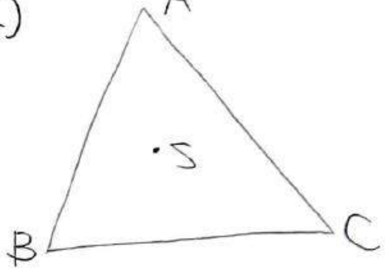
$S\left(\frac{4}{3}k+2, \frac{1}{3}k+1, \frac{1}{3}k+1\right)$

$\overrightarrow{AS}$

$=\overrightarrow{OS}-\overrightarrow{OA}$

$=\begin{pmatrix} \frac{4}{3}k+4 \\ \frac{1}{3}k+1 \\ \frac{1}{3}k+1 \end{pmatrix}$ //

(2)



$\overrightarrow{AS}=\alpha\overrightarrow{AB}+\beta\overrightarrow{AC}$

$=\alpha\begin{pmatrix}2\\1\\0\end{pmatrix}+\beta\begin{pmatrix}2\\0\\1\end{pmatrix}$

$=\begin{pmatrix}2\alpha+2\beta\\ \alpha \\ \beta\end{pmatrix}$ とする.

SがΔABCの内部または周にあるには

$\alpha \geqq 0,\ \beta \geqq 0,\ \alpha+\beta \leqq 1$

であればいいので

$\frac{1}{3}k+1 \geqq 0$

$\frac{4}{3}k+4 \leqq 2$

$\Downarrow$

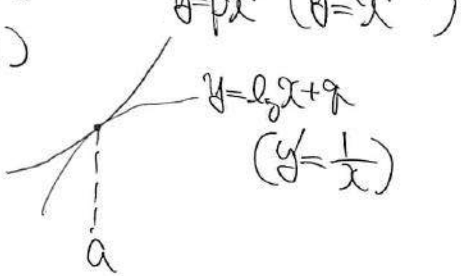
$0 \leqq \frac{1}{3}k+1 \leqq \frac{1}{2}$

$\iff -1 \leqq \frac{1}{3}k \leqq -\frac{1}{2}$

$\therefore -3 \leqq k \leqq -\frac{3}{2}$ //

[4]

(1)

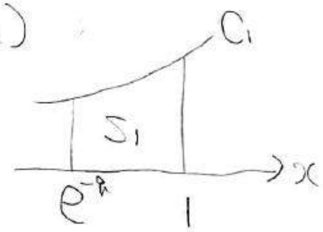


C_1, C_2が(a,b)で接するので

$\begin{cases} a^{\frac{1}{p}-1}=\frac{1}{a} \iff a^{\frac{1}{p}}=1 \\ pa^{\frac{1}{p}}=\log a+q \end{cases}$

$\therefore a=1,\ q=p$ //

(2)

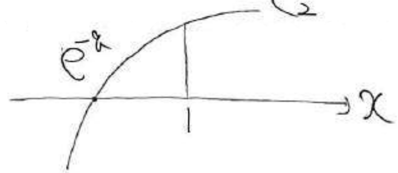


S_1

$=\int_{e^{-q}}^{1} px^{\frac{1}{p}}dx$

$=\left[\frac{p}{\frac{1}{p}+1}x^{\frac{1}{p}+1}\right]_{e^{-p}}^{1}$

$=\frac{p^2}{1+p}-\frac{p^2}{1+p}e^{-1-p}$ //



S_2

$=\int_{e^{-q}}^{1}(\log x+q)dx$

$=\left[x\log x-x+qx\right]_{e^{-q}}^{1}$

$=-1+q-(-qe^{-q}-e^{-q}+qe^{-q})$

$=p+e^{-p}-1$ //

(3)

$\frac{S_2}{S_1}$

$=\frac{p+e^{-p}-1}{\frac{p^2}{1+p}(1-e^{-1-p})}$

$=\frac{p^2-1+(p+1)e^{-p}}{p^2(1-e^{-1-p})}$

$\frac{S_2}{S_1} \geqq \frac{3}{4}$

$\iff \frac{p^2-1+(p+1)e^{-p}}{p^2(1-e^{-1-p})} \geqq \frac{3}{4}$

$\iff 4p^2-4+4(p+1)e^{-p} \geqq 3p^2(1-e^{-1-p})$

$\iff p^2-4+4(p+1)e^{-p}+3p^2e^{-1-p} \geqq 0$ …☆

左辺を$f(p)$とおく.

$f'(p)$

$=2p+4e^{-p}-4(p+1)e^{-p}+6pe^{-1-p}-3p^2e^{-1-p}$

$=pe^{-p-1}\{2e^{p+1}-4e-3(p-2)\}$

$g(p)=2e^{p+1}-4e-3(p-2)$

とおく.

$g'(p)=2e^{p+1}-3$

$\quad >2e-3$

$\quad >2\cdot 2.5-3=2>0$

$g(p)$ は $0<p<1$ において

単調増加.

$g(0)=-2e+6$

$\quad >-2\cdot 3+6=0$

より $0<p<1$ で $g(p)>0$.

$\therefore f'(p)>0$

$f(p)$ は $0<p<1$ において

単調増加.

$f(0)=-4+4=0$

より $0<p<1$ で $f(p)>0$

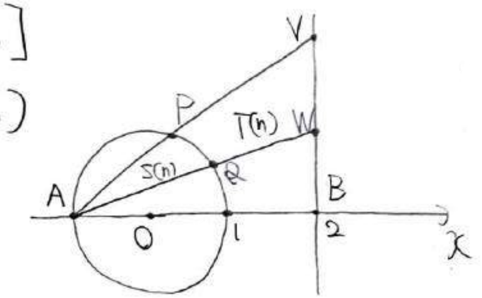
☆が成り立つので

$\dfrac{S_2}{S_1}\geqq\dfrac{3}{4}$

が成り立つ.

[5]

(1)



$\angle POB=\dfrac{\pi}{n}$ より

$\angle PAB=\dfrac{\pi}{2n}$.

同様に $\angle QOB=\dfrac{\pi}{2n}$ より

$\angle QAB=\dfrac{\pi}{4n}$.

これより

$VB=3\tan\dfrac{\pi}{2n}$

$WB=3\tan\dfrac{\pi}{4n}$

$\therefore VW=3\left(\tan\dfrac{\pi}{2n}-\tan\dfrac{\pi}{4n}\right)$

$n\{S(n)+T(n)\}$

$=n\cdot VW\times 3\times\dfrac{1}{2}$

$=\dfrac{9n}{2}\left(\tan\dfrac{\pi}{2n}-\tan\dfrac{\pi}{4n}\right)$

$\lim\limits_{n\to\infty}n\{S(n)+T(n)\}$

$=\lim\limits_{n\to\infty}\left\{\dfrac{2n}{\pi}\tan\dfrac{\pi}{2n}\times\dfrac{\pi}{2n}\times\dfrac{9n}{2}\right.$

$\quad\left.-\dfrac{4n}{\pi}\tan\dfrac{\pi}{4n}\times\dfrac{\pi}{4n}\times\dfrac{9n}{2}\right\}$

$=\dfrac{9}{4}\pi-\dfrac{9}{8}\pi$

$=\dfrac{9}{8}\pi$ //

(2)

$S(n)$

$=\triangle AOP+\pi\times\dfrac{\frac{\pi}{n}}{2\pi}$

$\quad-\triangle AOQ-\pi\times\dfrac{\frac{\pi}{2n}}{2\pi}$

$=\dfrac{1}{2}\sin\left(\pi-\dfrac{\pi}{n}\right)$

$\quad-\dfrac{1}{2}\sin\left(\pi-\dfrac{\pi}{2n}\right)+\dfrac{\pi}{4n}$

$=\dfrac{1}{2}\left(\sin\dfrac{\pi}{n}-\sin\dfrac{\pi}{2n}\right)+\dfrac{\pi}{4n}$

$\dfrac{S(n)+T(n)}{S(n)}$

$=\dfrac{\frac{9}{2}\left(\tan\frac{\pi}{2n}-\tan\frac{\pi}{4n}\right)}{\frac{1}{2}\sin\frac{\pi}{n}-\frac{1}{2}\sin\frac{\pi}{2n}+\frac{\pi}{4n}}$

$=\dfrac{9\left(\dfrac{\tan\frac{\pi}{2n}}{\frac{\pi}{2n}}-\dfrac{\tan\frac{\pi}{4n}}{\frac{\pi}{4n}}\times\dfrac{1}{2}\right)}{\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\times 2-\dfrac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}+\dfrac{\frac{\pi}{2n}}{\frac{\pi}{2n}}}$

$\to\dfrac{9\left(1-1\cdot\frac{1}{2}\right)}{2-1+1}=\dfrac{9}{4}\ (n\to\infty)$

$\therefore\lim\limits_{n\to\infty}\dfrac{T(n)}{S(n)}=\dfrac{9}{4}-1$

$=\dfrac{5}{4}$ //

[6]

(1)

$\omega=\cos\dfrac{2}{3}\pi+i\sin\dfrac{2}{3}\pi$

$\omega^2=\cos\dfrac{4}{3}\pi+i\sin\dfrac{4}{3}\pi$

$\quad=-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$

$AB=|1-\omega|$

$\quad=\left|\dfrac{3}{2}-\dfrac{\sqrt{3}}{2}i\right|$

$\quad=\sqrt{3}$

$BC=|\omega-\omega^2|$

$\quad=|\sqrt{3}i|$

$\quad=\sqrt{3}$

$CA=|\omega^2-1|$

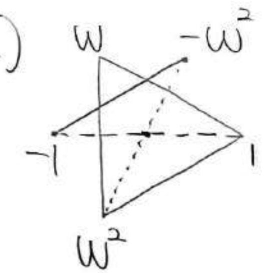
$\quad=\left|-\dfrac{3}{2}-\dfrac{\sqrt{3}}{2}i\right|$

$\quad=\sqrt{3}$

より $AB=BC=CA$ なので

$\triangle ABC$ は正三角形.

(2)



辺ACをOに関して対称に移動した線分。つまり-1と$-w^2$を結ぶ線分。

(3)

zが辺AB上を動くとき$z=z_1$、辺AC上を動くとき$z=z_2$とすると、E_1とE_2が共有点をもつのは

$$z_1^2=z_2^2$$

$\iff z_1=\pm z_2$

(i) $z_1=z_2$のとき

共有点は1のみ。

(ii) $z_1=-z_2$のとき

(2)を参考にすると辺ABと-1と$-w^2$を結ぶ交点となる。座標平面上で考えると

$$\begin{cases} y=-\frac{1}{\sqrt{3}}(x-1) \\ y=\frac{1}{\sqrt{3}}(x+1) \end{cases}$$

を連立して

$-x+1=x+1$

$\therefore x=0,\ y=\frac{1}{\sqrt{3}}$

共有点は$\frac{1}{\sqrt{3}}i$のみ。

(i)(ii)より求める共有点はE_1, E_2上の点を考えると注して

$1,\ -\frac{1}{3}$