2021 上智大 理工
(学部学科試験・共通テスト併用型)

$\boxed{1}$

(1) $\dfrac{x}{2}=\dfrac{1}{\cos\theta},\ \dfrac{y-1}{3}=\tan\theta.$

$1+\tan^2\theta=\dfrac{1}{\cos^2\theta}$ より

$1+\left(\dfrac{y-1}{3}\right)^2=\dfrac{x^2}{4}$

$\Leftrightarrow \dfrac{x^2}{4}-\dfrac{(y-1)^2}{9}=1$

頂点 $(\underline{\pm 2},\ \underline{1})$

焦点 $(\underline{\pm\sqrt{13}},\ \underline{1})$

漸近線 $y-1=\pm\dfrac{3}{2}x$

$\therefore y=\underline{\pm\dfrac{3}{2}}x+\underline{1}$

(2) $x=4$のとき

$4-\dfrac{(y-1)^2}{9}=1$

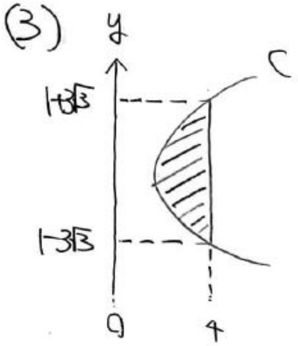
$\Leftrightarrow 27=(y-1)^2$

$\therefore y=1\pm 3\sqrt{3}$

Cと$x=4$は

$(4,\ \underline{1\pm 3\sqrt{3}})$で交わる.

(3)



求める体積は

$16\pi\cdot 6\sqrt{3}-\displaystyle\int_{1-3\sqrt{3}}^{1+3\sqrt{3}} x^2\pi\,dy$

$=96\sqrt{3}\pi-4\pi\displaystyle\int_{1-3\sqrt{3}}^{1+3\sqrt{3}}\left\{1+\dfrac{(y-1)^2}{9}\right\}dy$

$=96\sqrt{3}\pi-4\pi\left[y+\dfrac{(y-1)^3}{27}\right]_{1-3\sqrt{3}}^{1+3\sqrt{3}}$

$=96\sqrt{3}\pi-4\pi\{6\sqrt{3}+3\sqrt{3}-(-3\sqrt{3})\}$

$=\underline{48\sqrt{3}\pi}$

$\boxed{2}$

(i)

(a)

$f(x)=xe^{-x}$

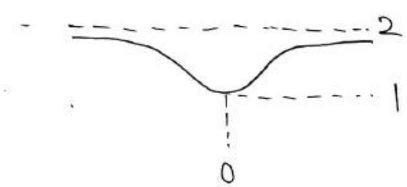
$f'(x)=e^{-x}-xe^{-x}$

$=e^{-x}(1-x)$

$\underline{\times}$

(b)

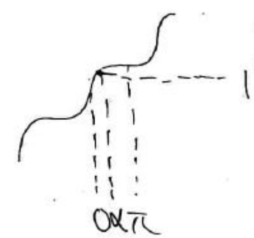
$f(x)=2-\dfrac{1}{x^2+1}$



$k=0$にすると(iii). $\underline{\bigcirc}$

(c) $f(x)=x+\sin x$

$f'(x)=1+\cos x\geqq 0$



$\underline{\bigcirc}$

(d) $f(x)=x\sin x$

$f'(x)=\sin x+x\cos x$

$f(x)$は正と負の値を取り続けるので $\underline{\times}$

(ii) あ…(a)

い…(b)

う…(b)

え…(a)

(iii) (A)…○

(B)…×

(C)…×

(D)…×

(2)

(i) $10^{\frac{k-1}{k}}<7<10^{\frac{k}{k+1}}$

$10^{k-1}<7^k \qquad 7^{k+1}<10^k$

$7\cdot 10^{k-1}<7^{k+1}<10^k$

これをみたすのは $\underline{k=6}$

(ii) $29<35\log_{10}7<30$

$\Leftrightarrow 10^{29}<7^{35}<10^{30}$ $\underline{30桁}$

$\boxed{3}$

(1)

α^2

$= (\omega+\omega^4)^2$

$= \omega^2+2\omega^5+\omega^8$

$= \omega^2+2+\omega^3$

$= 1+\omega+\omega^2+\omega^3+\omega^4$

$\quad +1-\alpha$

$= \dfrac{1-\omega^5}{1-\omega}+1-\alpha$

$= -\alpha+1 \cdots \underline{(k)}$ //

$\iff \alpha^2+\alpha-1=0$

$\therefore \alpha = \dfrac{-1\pm\sqrt{5}}{2}$

ところで

$\alpha = \omega+\omega^4$

$\quad = \omega+\overline{\omega}$

$\quad = 2\cos\frac{2\pi}{5} > 0$

より

$\alpha = \dfrac{-1+\sqrt{5}}{2}$ //

(2)

$\left|\frac{i}{2}-(-1)\right|$

$= \left|1+\frac{1}{2}i\right|$

$= \dfrac{\sqrt{5}}{2} = \alpha+\dfrac{1}{2} \cdots \underline{(m)}$ //

(3)

$|\omega^2-(-1)|$

$= \left|\cos\frac{4\pi}{5}+1+i\sin\frac{4\pi}{5}\right|$

$= \sqrt{\left(\cos\frac{4\pi}{5}+1\right)^2+\sin^2\frac{4\pi}{5}}$

$= \sqrt{2+2\cos\frac{4\pi}{5}}$

$= \sqrt{2+4\cos^2\frac{2\pi}{5}-2}$

$= \sqrt{\alpha^2}$

$= \alpha \cdots\cdots \underline{(c)}$ //

(4)

ω^2+1

$= \cos\frac{4\pi}{5}+1+i\sin\frac{4\pi}{5}$

$= 2\cos^2\frac{2\pi}{5}+2i\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}$

$= 2\cos\frac{2\pi}{5}\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)$

同様に

$\omega+1 = 2\cos\frac{\pi}{5}\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$

$\dfrac{\omega^2+1}{\omega+1}$

$= \dfrac{\cos\frac{2\pi}{5}}{\cos\frac{\pi}{5}}\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$

$= \dfrac{\alpha}{2}\cdot\dfrac{1}{-\cos\frac{4\pi}{5}}(\quad〃\quad)$

$= \dfrac{\alpha}{-(-\alpha-1)}(\quad〃\quad)$

$= \dfrac{1-\alpha^2}{1+\alpha}(\quad〃\quad)$

$= (1-\alpha)\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$

$r = -\alpha+1 \cdots \underline{(k)}$ //

$\theta = \frac{1}{5}\pi$ //

(5)

(3)より $|\omega^3-(-1)|=\alpha$

これより

$|z+1|=\alpha$

$\downarrow z=\frac{1}{x}$

$\left|\frac{1}{x}+1\right|=\alpha$

$\therefore |1+x|=\alpha|x|$

$(†)\cdots \underline{(c)}$ //

2乗して

$(x+1)(\overline{x}+1)=\alpha^2x\overline{x}$

$\iff (1-\alpha^2)x\overline{x}+x+\overline{x}+1=0$

$\iff x\overline{x}+\frac{1}{1-\alpha^2}x+\frac{1}{1-\alpha^2}\overline{x}+\frac{1}{1-\alpha^2}=0$

$\iff \left(x+\frac{1}{1-\alpha^2}\right)\left(\overline{x}+\frac{1}{1-\alpha^2}\right)=\left(\frac{1}{1-\alpha^2}\right)^2-\frac{1}{1-\alpha^2}$

$\iff \left(x+\frac{1}{\alpha}\right)\left(\overline{x}+\frac{1}{\alpha}\right)=\frac{1}{\alpha^2}-\frac{1}{\alpha}$

$\iff \left|x+\frac{1}{\alpha}\right|^2=\frac{1-\alpha}{\alpha^2}=1$

$\therefore \left|x+\frac{1}{\alpha}\right|=1$

$-\frac{1}{\alpha}=-\alpha-1 \cdots \underline{(l)}$ //

$\boxed{1=\alpha^2+\alpha \iff \frac{1}{\alpha}=\alpha+1}$

を中心とした半径$1 \cdots \underline{(a)}$ //

の円を描く.

$\boxed{4}$

(1) $x=0$ のとき P と O が同じ.

$\theta = \frac{\pi}{2} \cdots \underline{(d)}$

$x=1$ のとき P と G が同じ.

$\theta = \frac{\pi}{3} \cdots \underline{(c)}$

(2)

$\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}, \overrightarrow{OC}=\vec{c}$

とおく.

$\overrightarrow{OP} = x\overrightarrow{OG}$

$= x(\vec{a}+\vec{b}+\vec{c})$

$\overrightarrow{PA}\cdot\overrightarrow{PB}$

$= \{(1-x)\vec{a}-x\vec{b}-x\vec{c}\}\cdot\{-x\vec{a}+(1-x)\vec{b}-x\vec{c}\}$

$= -x(1-x)|\vec{a}|^2 - x(1-x)|\vec{b}|^2 + x^2|\vec{c}|^2$

$= (3x^2-2x)|\vec{a}|^2 = 0$

$\Leftrightarrow x = 0, \frac{2}{3}$

(3) $|\overrightarrow{PA}|^2 = |\overrightarrow{PB}|^2$

$= (1-x)^2|\vec{a}|^2 + x^2|\vec{b}|^2 + x^2|\vec{c}|^2$

$= (3x^2-2x+1)|\vec{a}|^2$

$y = \cos\theta$

$= \dfrac{\overrightarrow{PA}\cdot\overrightarrow{PB}}{|\overrightarrow{PA}||\overrightarrow{PB}|}$

$= \dfrac{3x^2-2x}{3x^2-2x+1}$

$= 1 - \dfrac{1}{3x^2-2x+1}$

$\dfrac{dy}{dx} = \dfrac{6x-2}{(3x^2-2x+1)^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{-2(27x^2-18x+1)}{(3x^2-2x+1)^3}$

$\dfrac{d^2y}{dx^2} = 0 \Leftrightarrow 27x^2-18x+1=0$

$\Leftrightarrow x = \dfrac{3\pm\sqrt{6}}{9}$

x	0	$\cdots$	$\frac{3-\sqrt{6}}{9}$	$\cdots$	$\frac{1}{3}$	$\cdots$	$\frac{3+\sqrt{6}}{9}$	$\cdots$	1
$\frac{dy}{dx}$		$-$	$-$	$-$	0	$+$	$+$	$+$	
$\frac{d^2y}{dx^2}$		$-$	0	$+$	$+$	$+$	0	$-$	
y	0	↘	$-\frac{1}{8}$	↘	$-\frac{1}{2}$	↗	$-\frac{1}{8}$	↗	$\frac{1}{2}$

$f\left(\frac{3\pm\sqrt{6}}{9}\right)$

$= 1 - \dfrac{1}{\frac{1}{9}(27x^2-18x+1)+\frac{8}{9}} = -\dfrac{1}{8}$