2020 早稲田 理工系

[I]

(1) $\alpha+\beta=-\gamma$

$\therefore |\alpha+\beta|=|-\gamma|=1$

原点をO, A(α), B(β), C(γ)

$\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, $\overrightarrow{OC}=\vec{c}$

とすると

$|\vec{a}|=1$, $|\vec{b}|=1$ $|\vec{a}+\vec{b}|=1$

$\therefore |\vec{a}+\vec{b}|^2=1+2\vec{a}\cdot\vec{b}+1=1$

$\therefore \vec{a}\cdot\vec{b}=-\frac{1}{2}$

$\therefore \angle AOB=120^\circ$

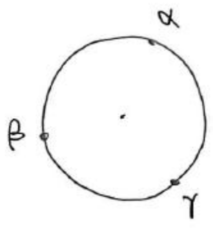
同様に $\angle BOC=\angle COA=120^\circ$

$\triangle AOB\equiv\triangle BOC\equiv\triangle COA$ より

$AB=BC=CA$

つまり α, β, γ を表す点が正三角形をなす.

(2)



$\beta=\alpha\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)$

$\gamma=\alpha\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)$

とおいて一般性を失わない.

$$\frac{\alpha\beta}{\gamma^2}+\frac{\beta\gamma}{\alpha^2}+\frac{\gamma\alpha}{\beta^2}$$

$$=\frac{\alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3}{\alpha^2\beta^2\gamma^2}$$

$$=\frac{\alpha^6+\alpha^6+\alpha^6}{\underbrace{\alpha^2\cdot\alpha^2\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)}_{\alpha^2\left(\cos\frac{8}{3}\pi+i\sin\frac{8}{3}\pi\right)}}$$

$$=\frac{3\alpha^6}{\alpha^6}=\underline{3}$$

(3)

(i) $n=3k+1$ (kは0以上の整数) のとき

$\alpha^n+\beta^n+\gamma^n$

$=\alpha^n+\alpha^n\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)^{3k+1}$

$\quad+\alpha^n\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)^{3k+1}$

$=\alpha^n+\alpha^n\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)$

$\quad+\alpha^n\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)$

$=\alpha^n\left(1-\frac{1}{2}-\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{\sqrt{3}}{2}i\right)$

$=0$

(ii) $n=3k+2$ のとき (kは0以上の整数)

$\alpha^n+\beta^n+\gamma^n$

$=\alpha^n+\alpha^n\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)^{3k+2}$

$\quad+\alpha^n\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)^{3k+2}$

$=\alpha^n+\alpha^n\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)$

$\quad+\alpha^n\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)$

$=0$

以上より $\alpha^n+\beta^n+\gamma^n=\underline{0}$

[II]

(1) CのPにおける接線は

$y=t(x-t)+\frac{1}{2}t^2$

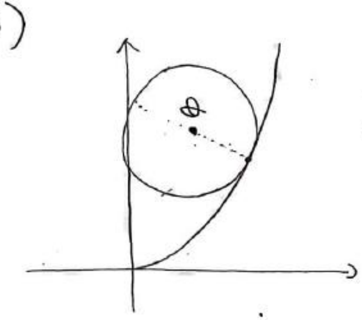
$=\underline{tx-\frac{1}{2}t^2}$

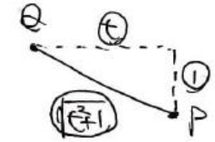
(2) CのPにおける法線は

$y=-\frac{1}{t}(x-t)+\frac{1}{2}t^2$

$=\underline{-\frac{1}{t}x+\frac{1}{2}t^2+1}$

(3)





$\underline{Y=\frac{1}{2}t^2+\frac{d}{\sqrt{t^2+1}}}$

(4) $t^2=u$ として

$Y=\frac{1}{2}u+\frac{d}{\sqrt{u+1}}=f(u)$ とおく

$\frac{df(u)}{du}=\frac{1}{2}-\frac{1}{2}(u+1)^{-\frac{3}{2}}d$

$=\frac{1}{2}\left\{1-\frac{d}{(u+1)^{\frac{3}{2}}}\right\}$

(i) $0<d\leqq 1$ のとき

$\frac{df(u)}{du}>0$

求める極小値は $f(0)=\underline{d}$

(ii) $d>1$ のとき

0	$0\cdots d^{\frac{2}{3}}-1\cdots$
$\frac{df(u)}{du}$	$-\ -\ 0\ +$
$f(u)$	↘ ↗

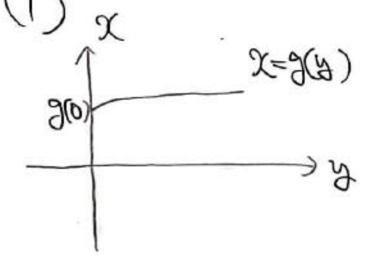
求める極小値は

$f(d^{\frac{2}{3}}-1)=\frac{1}{2}d^{\frac{2}{3}}-\frac{1}{2}+d^{\frac{2}{3}}$

$=\underline{\frac{3d^{\frac{2}{3}}-1}{2}}$

[III]

(1)



$$V=\int_0^{h(t)}\{g(y)\}^2\pi\,dy$$

↓ tで微分

$$\frac{dV}{dt}=\{g(h(t))\}^2\pi\cdot\frac{dh(t)}{dt}$$

$$\therefore \frac{v}{\frac{dh(t)}{dt}}=\{g(y)\}^2\pi$$

ここで $\frac{dh(t)}{dt}$ は定数なので

$g(y)$ は定数関数.

(2)

$$V=\int_0^{h(t)}\pi e^{2y}\,dy$$

$$=\left[\frac{\pi}{2}e^{2y}\right]_0^{h(t)}$$

$$vt=\frac{\pi}{2}\left(e^{2h(t)}-1\right)$$

$$\Longleftrightarrow \frac{2vt}{\pi}=e^{2h(t)}-1$$

$$\Longleftrightarrow e^{2h(t)}=\frac{2vt}{\pi}+1$$

$$\therefore h(t)=\frac{1}{2}\log\left(\frac{2vt}{\pi}+1\right)$$

[IV]

(1)

$P_1^{(2)}$

$$=\alpha+\beta^2\alpha+\beta^4\alpha+\cdots$$

$$=\frac{\alpha}{1-\beta^2}$$

$$=\frac{1-\beta}{1-\beta^2}=\underline{\frac{1}{1+\beta}}\ /\!/$$

(2)

$P_k^{(n)}$

$$=\beta^{k-1}\alpha+\beta^{n+k-1}\alpha+\cdots$$

$$=\frac{\beta^{k-1}\alpha}{1-\beta^n}$$

$$\therefore \frac{P_{k+1}^{(n)}}{P_k^{(n)}}=\underline{\beta}\ /\!/$$

(3)

$$q_k=\lim_{n\to\infty}P_k^{(n)}=\beta^{k-1}\alpha$$

$$E_m=\sum_{k=1}^{m}k\,q_k$$

$$=1\cdot\beta^0\cdot\alpha+2\beta\alpha+\cdots+m\beta^{m-1}\alpha$$

$$-)\ \beta E_m=\qquad \beta\alpha+\cdots+(m-1)\beta^{m-1}\alpha+m\beta^m\alpha$$

$$(1-\beta)E_m=\alpha+\beta\alpha+\cdots+\beta^{m-1}\alpha-m\beta^m\alpha$$

$$=\frac{\alpha}{1-\beta}-m\beta^m\alpha$$

$$\alpha E_m=1-m\beta^m\alpha$$

$$\therefore E_m=\frac{1-m\beta^m\alpha}{\alpha}$$

$$E=\lim_{m\to\infty}E_m$$

$$=\frac{1}{\alpha}$$

$$=\frac{a+b}{a}$$

$$=1+\frac{b}{a}\quad(a\geq b)$$

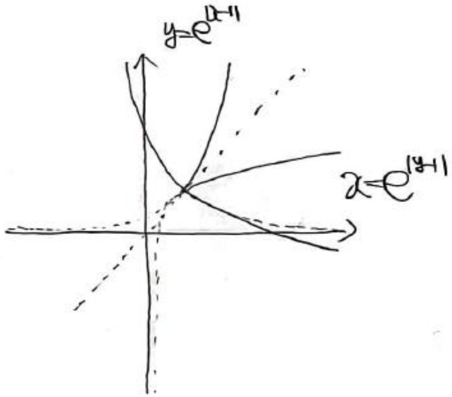
$\underline{a=b=1010\text{ のとき }E\text{ は最大}}$ //

[IV]

(1)

$y=e^{|x-1|}$

$=\begin{cases} e^{x-1} & (x\geqq 1) \\ e^{1-x} & (x<1) \end{cases}$



(2)

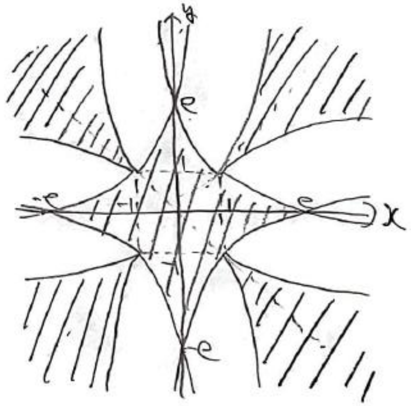
$-e^{|x-1|}\leqq y\leqq e^{|x-1|}$

$-e^{|x+1|}\leqq y\leqq e^{|x+1|}$

$-e^{|y-1|}\leqq x\leqq e^{|y-1|}$

$-e^{|y+1|}\leqq x\leqq e^{|y+1|}$

これらの共通範囲



(3), (4)は全員が得点とする措置だそうです.