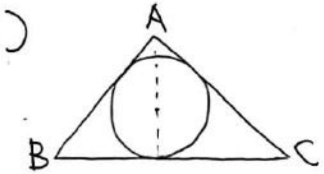
2020 筑波大

[1]〜[3] 数IIB

[4]〜[6] 数III

[1]

(1)



$\underset{1}{\triangle ABC} = \frac{r}{2}(\sqrt{2}+2+\sqrt{2})$

$\therefore r = \frac{1}{\sqrt{2}+1} = \underline{\sqrt{2}-1}$ //

(2)

連立して

$x^2+(ax-1)^2=1$

$\Leftrightarrow (1+a^2)x^2-2ax=0$

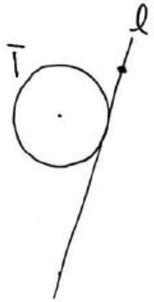
$\Leftrightarrow x\{(1+a^2)x-2a\}=0$

$x=0, \frac{2a}{1+a^2}$

$a\frac{2a}{1+a^2}-1=\frac{a^2-1}{1+a^2}$

より $\underline{E\left(\frac{2a}{1+a^2}, \frac{a^2-1}{a^2+1}\right)}$ //

(3)



円Tの中心$(0,\sqrt{2}-1)$と

$l: ax-y-1=0$の距離が$\sqrt{2}-1$

なので

$\frac{|-\sqrt{2}+1-1|}{\sqrt{a^2+1}}=\sqrt{2}-1$

$\Leftrightarrow \underbrace{\frac{\sqrt{2}}{\sqrt{2}-1}}_{2+\sqrt{2}}=\sqrt{a^2+1}$

2乗して

$a^2+1=6+4\sqrt{2}$

$\therefore a^2=5+4\sqrt{2}$

Eのy座標は

$\frac{4+4\sqrt{2}}{6+4\sqrt{2}}=\frac{2+2\sqrt{2}}{3+2\sqrt{2}}$

$=2(1+\sqrt{2})(3-2\sqrt{2})$

$=2(\sqrt{2}-1)$

Eを通りx軸に平行な直線は

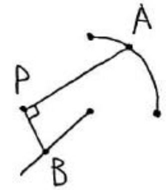
$y=2(\sqrt{2}-1)$, 円Tの中心が

$(0,\sqrt{2}-1)$, 半径が$\sqrt{2}-1$なので

円Tに接する.

[2]

(1)



$B(b,b)$　$(b\leqq 0)$とおくと

$\overrightarrow{PA}\cdot\overrightarrow{PB}$

$=\begin{pmatrix}\cos\theta+1\\ \sin\theta\end{pmatrix}\cdot\begin{pmatrix}b+1\\ b\end{pmatrix}=0$

$\Leftrightarrow (\cos\theta+1)(b+1)+\sin\theta\cdot b=0$

$\Leftrightarrow (\cos\theta+1+\sin\theta)b=-\cos\theta-1$

$0\leqq\theta\leqq\frac{\pi}{2}$ より $\cos\theta\geqq 0$

$\sin\theta\geqq 0$ より $\cos\theta+1+\sin\theta\geqq 1$.

$b=\frac{-\cos\theta-1}{\sin\theta+\cos\theta+1}$

$\underline{B\left(-\frac{\cos\theta+1}{\sin\theta+\cos\theta+1}, -\frac{\cos\theta+1}{\sin\theta+\cos\theta+1}\right)}$ //

(2)

線分ABの中点のx座標を

$f(\theta)$とすると

$f(\theta)$

$=\frac{1}{2}\left(\cos\theta-\frac{\cos\theta+1}{\sin\theta+\cos\theta+1}\right)$

$=\frac{1}{2}\cdot\frac{\sin\theta\cos\theta+\cos^2\theta-1}{\sin\theta+\cos\theta+1}\geqq 0$

$\sin\theta+\cos\theta+1\geqq 1$ より

$\sin\theta\cos\theta+\cos^2\theta-1\geqq 0$

$\Leftrightarrow \frac{1}{2}\sin 2\theta+\frac{1+\cos 2\theta}{2}-1\geqq 0$

$\Leftrightarrow \sin 2\theta+\cos 2\theta\geqq 1$

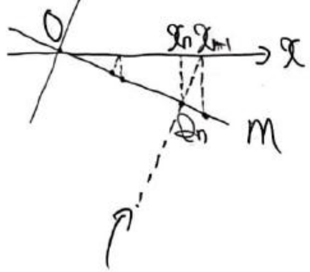
$\Leftrightarrow \sin\left(2\theta+\frac{\pi}{4}\right)\geqq\frac{1}{\sqrt{2}}$

$\frac{\pi}{4}\leqq 2\theta+\frac{\pi}{4}\leqq\frac{3}{4}\pi$

$\therefore \underline{0\leqq\theta\leqq\frac{\pi}{4}}$ //

[3]

(1)



$y=\sqrt{3}(x-x_n)-\frac{1}{\sqrt{3}}x_n$

$=\sqrt{3}x-\frac{4}{\sqrt{3}}x_n$

$\downarrow$ $(x_{n+1},0)$ 通る

$0=\sqrt{3}x_{n+1}-\frac{4}{\sqrt{3}}x_n$

$\Leftrightarrow x_{n+1}=\frac{4}{3}x_n$

$x_n=\left(\frac{4}{3}\right)^{n-1}$

(2)

$\overrightarrow{OP_n}=\begin{pmatrix} n \\ \sqrt{3}n \end{pmatrix}$ $\overrightarrow{OQ_n}=\begin{pmatrix} x_n \\ -\frac{x_n}{\sqrt{3}} \end{pmatrix}$

より

a_n

$=\frac{1}{2}\left|n\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{4}{3}\right)^{n-1}-\sqrt{3}n\left(\frac{4}{3}\right)^{n-1}\right|$

$=\frac{n}{2}\left(\frac{4}{3}\right)^{n-1}\left|-\frac{1}{\sqrt{3}}-\sqrt{3}\right|$

$=\frac{2n}{\sqrt{3}}\left(\frac{4}{3}\right)^{n-1}$

$=\frac{2\sqrt{3}n}{3}\left(\frac{4}{3}\right)^{n-1}$

(3)

$S_n=\frac{2\sqrt{3}}{3}\cdot 1+\cdots+\frac{2\sqrt{3}n}{3}\left(\frac{4}{3}\right)^{n-1}$

$-\underline{)\ \frac{4}{3}S_n=\qquad \frac{2\sqrt{3}}{3}\cdot\frac{4}{3}+\cdots+\frac{2\sqrt{3}(n-1)}{3}\left(\frac{4}{3}\right)^{n-1}+\frac{2\sqrt{3}n}{3}\left(\frac{4}{3}\right)^{n}}$

$-\frac{1}{3}S_n=\frac{2\sqrt{3}}{3}+\frac{2\sqrt{3}}{3}\cdot\frac{4}{3}+\cdots+\frac{2\sqrt{3}}{3}\left(\frac{4}{3}\right)^{n-1}-\frac{2\sqrt{3}n}{3}\left(\frac{4}{3}\right)^{n}$

$=\frac{\frac{2\sqrt{3}}{3}-\frac{2\sqrt{3}}{3}\left(\frac{4}{3}\right)^n}{1-\frac{4}{3}}-\frac{2\sqrt{3}n}{3}\left(\frac{4}{3}\right)^n$

$=2\sqrt{3}\left(\frac{4}{3}\right)^n-2\sqrt{3}-\frac{2\sqrt{3}n}{3}\left(\frac{4}{3}\right)^n$

$S_n=-6\sqrt{3}\left(\frac{4}{3}\right)^n+6\sqrt{3}+2\sqrt{3}n\left(\frac{4}{3}\right)^n$

$=2\sqrt{3}\left\{(n-3)\left(\frac{4}{3}\right)^n+3\right\}$

[4]

(1)

I_1

$=\left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}=\frac{1}{2}$

I_2

$=\int_0^{\frac{\pi}{4}}\sin\theta(2\cos^2\theta-1)d\theta$

$=\left[-\frac{2}{3}\cos^3\theta+\cos\theta\right]_0^{\frac{\pi}{4}}$

$=-\frac{2}{3}\frac{2\sqrt{2}}{8}+\frac{\sqrt{2}}{2}-\frac{1}{3}$

$=\frac{\sqrt{2}-1}{3}$

I_3

$=\int_0^{\frac{\pi}{4}}\sin^2\theta(2\cos^2\theta-1)d\theta$

$=\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\sin^2 2\theta-\frac{1-\cos 2\theta}{2}\right)d\theta$

$=\int_0^{\frac{\pi}{4}}\left(\frac{1-\cos 4\theta}{4}-\frac{1-\cos 2\theta}{2}\right)d\theta$

$=\int_0^{\frac{\pi}{4}}\left(-\frac{1}{4}\cos 4\theta+\frac{1}{2}\cos 2\theta-\frac{1}{4}\right)d\theta$

$=\left[-\frac{1}{16}\sin 4\theta+\frac{1}{4}\sin 2\theta-\frac{\theta}{4}\right]_0^{\frac{\pi}{4}}$

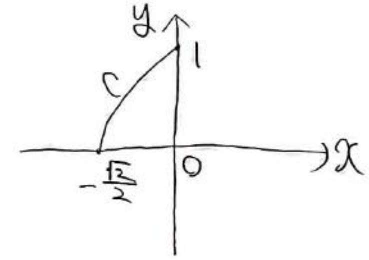
$=\frac{1}{4}-\frac{\pi}{16}$

(2)

$\frac{dx}{d\theta}=\cos\theta$ $\frac{dy}{d\theta}=2\cos 2\theta$

$\frac{dy}{dx}$

$=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{2\cos 2\theta}{\cos\theta}\geqq 0$



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$\int_0^1 x^2\pi dy$

$=\int_0^{\frac{\pi}{4}}\left(\sin\theta-\frac{\sqrt{2}}{2}\right)^2\pi\frac{dy}{d\theta}d\theta$

$=\pi\int_0^{\frac{\pi}{4}}\left(\sin^2\theta-\sqrt{2}\sin\theta+\frac{1}{2}\right)2\cos 2\theta\, d\theta$

$=\pi\int_0^{\frac{\pi}{4}}\left(2\sin^2\theta-2\sqrt{2}\sin\theta+1\right)\cos 2\theta\, d\theta$

$=\pi\left(2I_3-2\sqrt{2}I_2+I_1\right)$

$=\pi\left(\frac{1}{2}-\frac{\pi}{8}-\frac{4-2\sqrt{2}}{3}+\frac{1}{2}\right)$

$=\pi\left(\frac{2\sqrt{2}}{3}-\frac{1}{3}-\frac{\pi}{8}\right)$

[5]

(1)

$a_{n+1}-1=\frac{1}{2-a_n}-1$

$=\frac{a_n-1}{2-a_n}$

$b_n=a_n-1$ とおくと

$b_{n+1}=\frac{b_n}{1-b_n}$

$$b_1=\frac{c}{1+c}-1=\frac{-1}{1+c}\neq 0$$

漸化式の形より帰納的に

$b_n\neq 0$. 両辺逆数をとると

$$\frac{1}{b_{n+1}}=\frac{1}{b_n}-1$$

これより

$$\frac{1}{b_n}=\frac{1}{b_1}+(n-1)(-1)$$
$$=-1-c-n+1$$
$$=-n-c$$

$$\therefore\ b_n=\frac{1}{-n-c}$$

$$a_n=b_n+1$$
$$=\frac{1}{-n-c}+\frac{-n-c}{-n-c}$$
$$=\frac{n+c-1}{n+c}$$

これより

$$a_2=\frac{c+1}{2+c}\quad a_3=\frac{c+2}{3+c}$$

(2)

$$a_n=\frac{n+c-1}{n+c}$$

(3)

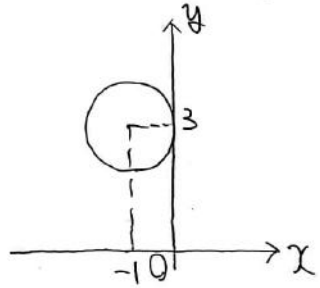
$$\sum_{n=1}^{\infty}\left(\frac{a_{n+1}}{a_n}-1\right)$$
$$=\sum_{n=1}^{\infty}\left(\frac{n+c}{n+1+c}\cdot\frac{n+c}{n+c-1}-1\right)$$
$$=\sum_{n=1}^{\infty}\left\{\frac{(n+c)^2-(n+c)^2+1}{(n+c)^2-1}\right\}$$
$$=\sum_{n=1}^{\infty}\frac{1}{(n+c-1)(n+c+1)}$$
$$=\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{n+c-1}-\frac{1}{n+c+1}\right)$$
$$=\lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{2}\left(\frac{1}{n+c-1}-\frac{1}{n+c+1}\right)$$
$$=\lim_{m\to\infty}\frac{1}{2}\left\{\left(\frac{1}{c}-\frac{1}{c+2}\right)+\left(\frac{1}{c+1}-\frac{1}{c+3}\right)+\left(\frac{1}{c+2}-\frac{1}{c+4}\right)+\cdots+\left(\frac{1}{m+c-2}-\frac{1}{m+c}\right)+\left(\frac{1}{m+c-1}-\frac{1}{m+c+1}\right)\right\}$$
$$=\lim_{m\to\infty}\frac{1}{2}\left(\frac{1}{c}+\frac{1}{c+1}-\frac{1}{m+c}-\frac{1}{m+c+1}\right)$$
$$=\frac{1}{2}\left(\frac{1}{c}+\frac{1}{c+1}\right)$$
$$=\frac{2c+1}{2c(c+1)}$$

[6]

(1)

$$z\bar{z}+(1+3i)z+(1-3i)\bar{z}+9$$
$$=\{z+(1-3i)\}\{\bar{z}+(1+3i)\}-1=0$$
$$\Leftrightarrow (z+1-3i)(\overline{z+1-3i})=1$$
$$\Leftrightarrow |z+1-3i|^2=1$$
$$\therefore\ |z+1-3i|=1$$

zは$-1+3i$を中心とする半径1の円.

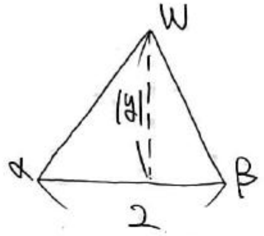


(2)

$$\begin{cases} w=x+yi \\ x=\cos\theta-1 \\ y=\sin\theta+3 \end{cases}$$ とおける.

$\alpha=w+\bar{w}-1$, $\beta=w+\bar{w}+1$

$=2\cos\theta-3$ $\quad=2\cos\theta-1$



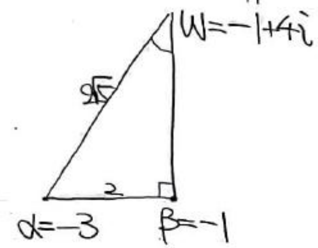
$\triangle PAB$

$=2\times|y|\times\frac{1}{2}$

$=|\sin\theta+3|=\sin\theta+3$

$\sin\theta=1$ のとき$\triangle PAB$は最大.

このとき $w=-1+4i$.



$$\sin\angle APB=\frac{1}{\sqrt{5}}$$

正弦定理より外接円の半径Rは

$$2R=\frac{2}{\sin\angle APB}=2\sqrt{5}$$

$$\therefore\ R=\sqrt{5}$$

外接円の中心は各辺の垂直二等分線の交点から

$$-2+2i$$