2020 上智大 理工

$\boxed{1}$

(1)

$$P_1(\cos\theta) = \frac{\sin 2\theta}{\sin\theta} = 2\cos\theta$$

$$\therefore P_1(x) = \underline{2}x + \underline{0}$$

$$P_2(\cos\theta) = \frac{\sin 3\theta}{\sin\theta} = 3 - 4\underbrace{\sin^2\theta}_{1-\cos^2\theta} = 4\cos^2\theta - 1$$

$$\therefore P_2(x) = \underline{4}x^2 + \underline{0}\cdot x \underline{-1}$$

$$P_3(\cos\theta) = \frac{\sin 4\theta}{\sin\theta} \quad \leftarrow 2\sin 2\theta\cos 2\theta$$

$$= 4\cos\theta\underbrace{\cos 2\theta}_{2\cos^2\theta - 1} = 8\cos^3\theta - 4\cos\theta$$

$$\therefore P_3(x) = \underline{8}x^3 + \underline{0}\cdot x^2 \underline{-4}x + \underline{0}$$

(2)

$$P_{n+1}(\cos\theta) = \frac{\sin(n+2)\theta}{\sin\theta}$$

$$= \frac{\sin(n+1)\theta\cos\theta + \cos(n+1)\theta\sin\theta}{\sin\theta}$$

積和変換

$$= P_n(\cos\theta)\cos\theta + \frac{\frac{1}{2}\{\sin(n+2)\theta - \sin n\theta\}}{\sin\theta}$$

$$= P_n(\cos\theta)\cos\theta + \frac{1}{2}P_{n+1}(\cos\theta) - \frac{1}{2}P_{n-1}(\cos\theta)$$

$$\Leftrightarrow \frac{1}{2}P_{n+1}(\cos\theta) = P_n(\cos\theta)\cos\theta - \frac{1}{2}P_{n-1}(\cos\theta)$$

$$\therefore P_{n+1}(\cos\theta) = 2P_n(\cos\theta)\cos\theta - P_{n-1}(\cos\theta)$$

$$\therefore P_{n+1}(x) = \underline{2x}P_n(x) + \underline{(-1)}P_{n-1}(x)$$

(3)

$$P_4(x) = 2xP_3(x) - P_2(x) = 2x(8x^3 - 4x) - (4x^2 - 1) = \underline{16x^4 - 12x^2 + 1}$$

$$P_4(1) = \underline{5}$$

(4)

$$P_4(\cos\theta) = \frac{\sin 5\theta}{\sin\theta} = 0$$

$$\min\theta = \underline{\frac{1}{5}\pi}$$

(5)

$$P_4(\cos\theta) = \frac{\sin 5\theta}{\sin\theta} = 0$$

$$\therefore \theta = \frac{k}{5}\pi \ (k = 1, 2, 3, 4)$$

$P_4(x) = 0$ の解が

$$x = \cos\frac{k}{5}\pi \ (k = 1, 2, 3, 4)$$

$$P_4(x) = 16\left(x - \cos\frac{\pi}{5}\right)\left(x - \cos\frac{2}{5}\pi\right)\left(x - \cos\frac{3}{5}\pi\right)\left(x - \cos\frac{4}{5}\pi\right)$$

↓ $x = 1$ を代入

$$P_4(1) = 16\left(1 - \cos\frac{\pi}{5}\right)\left(1 - \cos\frac{2}{5}\pi\right)\left(1 - \cos\frac{3}{5}\pi\right)\left(1 - \cos\frac{4}{5}\pi\right) = 5$$

より

$$\left(1 - \cos\frac{\pi}{5}\right)\left(1 - \cos\frac{2}{5}\pi\right)\left(1 - \cos\frac{3}{5}\pi\right)\left(1 - \cos\frac{4}{5}\pi\right) = \underline{\frac{5}{16}}$$

$\boxed{2}$

$$f'(x) = \frac{-ae^{-x} + 2e^{2x}}{ae^{-x} + e^{2x}} - e^{-x} = \frac{2e^{2x} - e^{x} - ae^{-x} - ae^{-2x}}{ae^{-x} + e^{2x}}$$

必要条件

$$f'(\log 2) = \frac{8 - 2 - \frac{1}{2}a - \frac{1}{4}a}{\frac{1}{2}a + 4} = \frac{24 - 3a}{2a + 16} = 0$$

$$\therefore a = 8$$

逆に $a = 8$ のとき

$$f'(x) = \frac{2e^{4x} - e^{3x} - 8e^{x} - 8}{8e^{x} + e^{3x}}$$

2	−1	0	−8	−8	
	4	6	12	+8	$\boxed{2}$
2	3	6	4		

$$= \frac{(e^{x} - 2)(2e^{3x} + 3e^{2x} + 6e^{x} + 4)}{e^{x}(8 + e^{2x})}$$

$x = \log 2$ で $f'(x)$ の符号が負から正になる. $\therefore a = \underline{8}$

(2) (i)

$(0, 2\log_2 3+1)$における接線は

$y=\frac{2-1-8-8}{8+1}x+2\log_2 3+1$

$=-\frac{5}{3}x+2\log_2 3+1$

$\downarrow$ $(p, 0)$通る

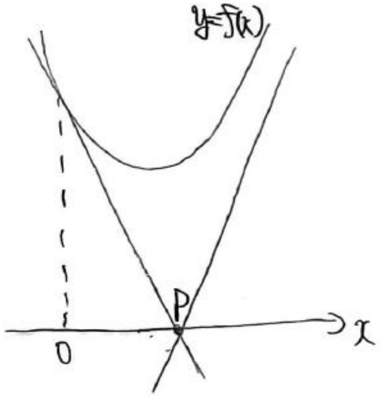
$0=-\frac{5}{3}p+2\log_2 3+1$

$\Leftrightarrow 5p=6\log_2 3+3$

$\therefore p=\frac{3}{5}+\frac{6}{5}\log_2 3$ //

(ii) 極値 $f(\log_2 2)=\log_2 8+\frac{1}{2}$

$=3\log_2 2+\frac{1}{2}$



$y=f(x)$の漸近線の傾きは2より

$y=m(x-p)$が$y=f(x)$の下側にあるには

$\underline{\frac{5}{3}}<m\underline{\leq 2}$ //

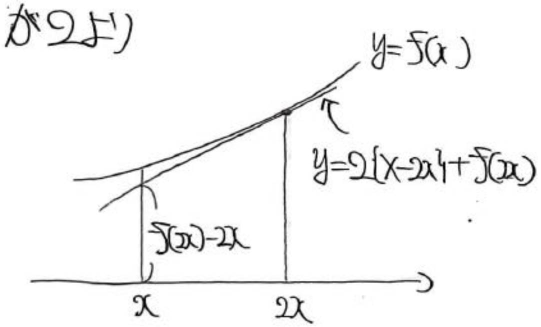
(3) $\int_x^{2x} f(t)dt=S$ とおく.

$f'(x)>0$より …※

$S<\square$

$=\{f(x)+f(2x)\}\frac{x}{2}$

また $y=f(x)$の漸近線の傾きが2より



$S>\square$

$=\{f(2x)-2x+f(2x)\}\frac{x}{2}$

$=xf(2x)-x^2$

以上より

$\frac{1}{x}f(2x)-1<\frac{S}{x^2}<\frac{f(x)+f(2x)}{2x}$

ここで

$\lim_{x\to\infty}\left\{\frac{1}{x}f(2x)-1\right\}$

$=\lim_{x\to\infty}\left\{\frac{1}{x}\log_2(8e^{-2x}+e^{6x})+e^{-2x}-1\right\}$

$=\lim_{x\to\infty}\left\{\frac{1}{x}\{4x+\log_2(8e^{-6x}+1)\}-1\right\}$

$=3$

$\lim_{x\to\infty}\frac{f(x)+f(2x)}{2x}$

$=\lim_{x\to\infty}\frac{\log_2 e^{2x}(8e^{-2x}+1)+e^{-x}+\log_2 e^{4x}(8e^{-6x}+1)+e^{-2x}}{2x}$

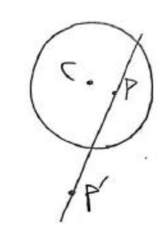
$=\lim_{x\to\infty}\frac{6x+\log_2(8e^{-2x}+1)+\log_2(8e^{-6x}+1)+e^{-x}+e^{-2x}}{2x}$

$=3$

以上より

$\lim_{x\to\infty}\frac{1}{x^2}\int_x^{2x}f(t)dt=\underline{3}$ //

$\boxed{3}$



直線PP': $\begin{cases} x=a+2t \\ y=b \\ z=-t \end{cases}$

$P(a+2p, b, -p)$とおく.

PはS上にあるので

$(a+2p)^2+b^2+(-p-2)^2=1$ …①

また $\overrightarrow{CP}\perp\begin{pmatrix}2\\0\\-1\end{pmatrix}$より

$\begin{pmatrix}a+2p\\b\\-p-2\end{pmatrix}\cdot\begin{pmatrix}2\\0\\-1\end{pmatrix}=2a+5p+2=0$

$\Leftrightarrow p=-\frac{2}{5}(a+1)$ …②

①,②より

$\{a-\frac{4}{5}(a+1)\}^2+b^2+\{\frac{2}{5}a-\frac{8}{5}\}^2=1$

$\Leftrightarrow \{\frac{1}{5}(a-4)\}^2+b^2+\{\frac{2}{5}(a-4)\}^2=1$

$\Leftrightarrow \frac{5}{25}(a-4)^2+b^2=1$

$\Leftrightarrow \frac{(a-4)^2}{5}+b^2=1$

長軸が$\underline{2\sqrt{5}}$, 短軸が$\underline{2}$ //

展開して

$a^2-8a+16+5b^2=5$

$\Leftrightarrow \underline{a^2}+\underline{5b^2}-\underline{8a}+\underline{0\cdot b}+\underline{11}=0$

(2)

$\overrightarrow{PP'}=\begin{pmatrix}2\\0\\-1\end{pmatrix}p$

$|\overrightarrow{PP'}|=|\sqrt{5}p|$

$=\left|\sqrt{5}\left\{-\frac{2}{5}(a+1)\right\}\right|$

$\frac{(a-4)^2}{5}+b^2=1$

aの定義域は $4-\sqrt{5}\leqq a\leqq 4+\sqrt{5}$.

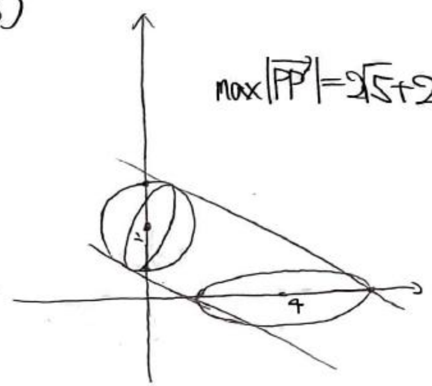
$a=4-\sqrt{5}$のとき

$\min|\overrightarrow{PP'}|=\frac{2\sqrt{5}}{5}(5-\sqrt{5})$

$=\underline{2\sqrt{5}-2}$

(3)

$\max|\overrightarrow{PP'}|=2\sqrt{5}+2$



求める体積は底面積がπで高さ$2\sqrt{5}$の円柱の体積と同じ.

$\pi\times 2\sqrt{5}=\underline{2\sqrt{5}\pi}$

$\boxed{4}$

(1)

P(3回でBが空)

$=\frac{1}{8}$

P(4回でBが空)

$=P(???B)$

$={}_3C_1\left(\frac{1}{2}\right)^4=\frac{3}{16}$

P(5回でBが空)

$={}_4C_2\left(\frac{1}{2}\right)^5=\frac{6}{32}$

P(6回でBが空)

$={}_5C_2\left(\frac{1}{2}\right)^6=\frac{10}{64}$

すべて足すと

$\frac{8+12+12+10}{64}=\frac{42}{64}=\underline{\frac{21}{32}}$

(2)

$A(0,6), B(6,0)$

と座標設定する.

$n((0,0)で会う)$ ＊(6,6)と同じ

$=1$

$n((1,1)で会う)$ ＊(5,5)と同じ

$={}_6C_1\cdot{}_6C_1=36$

$n((2,2)で会う)$ ＊(4,4)と同じ

$={}_6C_2\cdot{}_6C_2=225$

以上より

$2\times(1+36+225)=\underline{524}$