2020 日本医科大

[I]

$$\int_{-1}^{1}(3ax^2+2x+b)dx$$
$$=2\int_0^1(3ax^2+b)dx$$
$$=2\left[ax^3+bx\right]_0^1$$
$$=2a+2b=0 \iff b=-a$$

$$\int_{-1}^{1}(3ax^2+2x+b)^2dx$$
$$=2\int_0^1(9a^2x^4+4x^2+b^2+6abx^2)dx$$
$$=2\left[\frac{9a^2}{5}x^5+\frac{4}{3}x^3+b^2x+2abx^3\right]_0^1$$
$$=2\left(\frac{9a^2}{5}+\frac{4}{3}+b^2+2ab\right)=6$$
$$\iff \frac{9a^2}{5}+b^2+2ab=\frac{5}{3}$$
$$\downarrow b=-a$$
$$\frac{9}{5}a^2+a^2-2a^2=\frac{5}{3}$$
$$\iff \frac{4}{5}a^2=\frac{5}{3}$$
$$\iff a^2=\frac{25}{12}$$
$$\therefore a=\frac{5}{2\sqrt{3}}=\frac{5\sqrt{3}}{6}\ (a>0)$$

$$b=-\frac{5\sqrt{3}}{6}$$

$$\int_0^1\cos\left(\frac{\pi x}{2}\right)dx$$
$$=\left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^1$$
$$=\frac{2}{\pi}$$

$$\int_0^1\cos^2\left(\frac{\pi x}{2}\right)dx$$
$$=\int_0^1\frac{1+\cos\pi x}{2}dx$$
$$=\left[\frac{1}{2}x+\frac{1}{2\pi}\sin\pi x\right]_0^1$$
$$=\frac{1}{2}$$

$$\int_0^1 x^2\cos\left(\frac{\pi}{2}x\right)dx$$
$$=\left[\frac{2}{\pi}x^2\sin\left(\frac{\pi}{2}x\right)+\frac{8}{\pi^2}x\cos\left(\frac{\pi}{2}x\right)-\frac{16}{\pi^3}\sin\left(\frac{\pi}{2}x\right)\right]_0^1$$

$x^2\cos\left(\frac{\pi}{2}x\right)+\frac{4}{\pi}x\sin\left(\frac{\pi}{2}x\right)-\frac{4}{\pi}x\sin\left(\frac{\pi}{2}x\right)$

微分したもの $+\frac{8}{\pi^2}\cos\left(\frac{\pi}{2}x\right)-\frac{8}{\pi^2}\cos\left(\frac{\pi}{2}x\right)$

$$=\frac{2}{\pi}-\frac{16}{\pi^3}$$

$$\int_{-1}^{1}\left\{\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)-(pf(x)+q)\right\}^2dx$$
$$=\int_{-1}^{1}\left\{\frac{\pi^2}{4}\cos^2\left(\frac{\pi x}{2}\right)+p^2f(x)^2+2pqf(x)+q^2-\pi\cos\left(\frac{\pi x}{2}\right)(pf(x)+q)\right\}dx$$
$$=2\int_0^1\left\{\frac{\pi^2}{4}\cos^2\left(\frac{\pi x}{2}\right)+q^2-p\pi\cos\left(\frac{\pi x}{2}\right)(3ax^2+b)\right\}dx$$
$$+6p^2-q\pi\cdot 2\cdot\frac{2}{\pi}$$
$$=\frac{\pi^2}{2}\cdot\frac{1}{2}+2q^2$$
$$-2p\pi\cdot 3a\left(\frac{2}{\pi}-\frac{16}{\pi^3}\right)$$
$$-2p\pi\cdot b\cdot\frac{2}{\pi}+6p^2-4q$$
$$=6p^2-6ap\left(2-\frac{16}{\pi^2}\right)-4\underset{-a}{b}p$$
$$+2q^2-4q+\frac{\pi^2}{4}$$
$$=6p^2-8ap+\frac{96}{\pi^2}ap$$
$$+2q^2-4q+\frac{\pi^2}{4}$$
$$=6\left\{p^2+\left(\frac{16}{\pi^2}a-\frac{4}{3}a\right)p\right\}$$
$$+2(q-1)^2+\frac{\pi^2}{4}-2$$

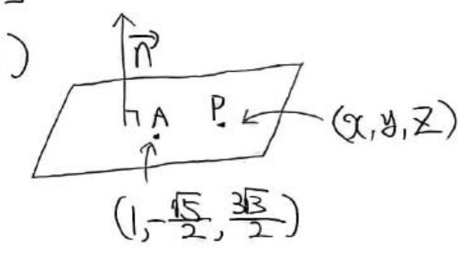
以上より

$$p=-\frac{8}{\pi^2}a+\frac{2}{3}a$$
$$=-\frac{8}{\pi^2}\cdot\frac{5\sqrt{3}}{6}+\frac{2}{3}\cdot\frac{5\sqrt{3}}{6}$$
$$=\frac{5\sqrt{3}}{9}-\frac{20\sqrt{3}}{3\pi^2}$$

$q=1$ のとき最小.

[II]

(1)



図のように α 上の P をおくと

$$\overrightarrow{AP}\cdot\vec{n}$$
$$=\begin{pmatrix}x-1\\ y+\frac{\sqrt{5}}{2}\\ z-\frac{3\sqrt{3}}{2}\end{pmatrix}\cdot\begin{pmatrix}-2\\ \sqrt{5}\\ \sqrt{3}\end{pmatrix}$$
$$=-2x+2+\sqrt{5}y+\frac{5}{2}+\sqrt{3}z-\frac{9}{2}=0$$
$$\iff -2x+\sqrt{5}y+\sqrt{3}z=0$$

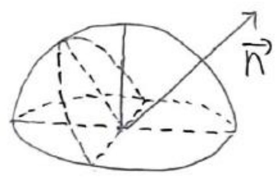
(2) なす角をθとすると

$$\begin{pmatrix}-2\\ \sqrt{5}\\ \sqrt{3}\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=\sqrt{4+5+3}\cdot 1\cdot\cos\theta$$

$\Leftrightarrow \sqrt{3}=2\sqrt{3}\cos\theta$

$\therefore \cos\theta=\frac{1}{2}$　$\theta=60^\circ$ //

(3)



平面αは原点を通る.

ベクトル(0,0,1)と$\vec{n}$のなす角が60°なので, 平面αとベクトル(0,0,1)のなす角が30°.

(0,0,3)を含む方の体積は球の体積の$\frac{120^\circ}{360^\circ}=\frac{1}{3}$である.

求める体積は

$\frac{4}{3}\pi\cdot 3^3\cdot\frac{1}{3}=12\pi$ //

[III]

(1) $x=\frac{k}{1+t^2k^2}>0$

$y=\frac{tk^2}{1+t^2k^2}$

$1+t^2k^2=\frac{k}{x}$

$\Leftrightarrow t^2=\frac{1}{kx}-\frac{1}{k^2}$

$(1+t^2k^2)y=tk^2$

$\Leftrightarrow \frac{k}{x}y=tk^2$

$\Leftrightarrow \frac{y}{x}=tk$

↓2乗

$\frac{y^2}{x^2}=\left(\frac{1}{kx}-\frac{1}{k^2}\right)k^2$

$\Leftrightarrow y^2=kx-x^2$

$\Leftrightarrow x^2-kx+y^2=0$

$\therefore \left(x-\frac{k}{2}\right)^2+y^2=\frac{k^2}{4}$

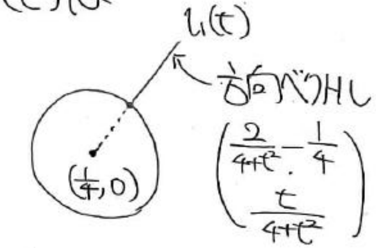
点$\left(\frac{k}{2},0\right)$が中心, 半径$\frac{k}{2}$の円 //

(2)

$T_t\left(\frac{1}{2}\right)=\left(\frac{\frac{1}{2}}{1+\frac{t^2}{4}},\frac{\frac{t}{4}}{1+\frac{t^2}{4}}\right)$

$=\left(\frac{2}{4+t^2},\frac{t}{4+t^2}\right)$

における$\left(x-\frac{1}{4}\right)^2+y^2=\frac{1}{16}$の法線$l_1(t)$は



$$\frac{x-\frac{1}{4}}{\frac{2}{4+t^2}-\frac{1}{4}}=\frac{y}{\frac{t}{4+t^2}}$$

$\Leftrightarrow \left(x-\frac{1}{4}\right)\frac{t}{4+t^2}=y\left(\frac{2}{4+t^2}-\frac{1}{4}\right)$

$\Leftrightarrow \left(x-\frac{1}{4}\right)t=y\left(2-\frac{4+t^2}{4}\right)$

$\Leftrightarrow tx-\frac{t}{4}=\left(1-\frac{t^2}{4}\right)y$

$\Leftrightarrow 4tx+(t^2-4)y-t=0$ //

同様に$l_2(t)$は

$$\frac{x-\frac{1}{2}}{\frac{1}{1+t^2}-\frac{1}{2}}=\frac{y}{\frac{t}{1+t^2}}$$

$\Leftrightarrow \left(x-\frac{1}{2}\right)\frac{t}{1+t^2}=y\left(\frac{1}{1+t^2}-\frac{1}{2}\right)$

$\Leftrightarrow \left(x-\frac{1}{2}\right)t=y\left(1-\frac{1+t^2}{2}\right)$

$\Leftrightarrow tx-\frac{t}{2}=y\left(\frac{1}{2}-\frac{t^2}{2}\right)$

$\Leftrightarrow 2tx+(t^2-1)y-t=0$ //

※$l_k(t)$を出してから$t=1,2$を代入した方が良かったです.

(3) $l_1(t)$と$l_2(t)$の交点は

$4tx+(t^2-4)y-t=0$

$-)\ 4tx+(2t^2-2)y-2t=0$

$(-t^2-2)y+t=0$

$\therefore y=\frac{t}{t^2+2}$

$2tx=(1-t^2)\frac{t}{t^2+2}+t$

$\Leftrightarrow x=\frac{1-t^2}{2t^2+4}+\frac{1}{2}=\frac{3}{2t^2+4}$

$\therefore P(t)\left(\frac{3}{2t^2+4},\frac{t}{t^2+2}\right)$ //

(4) $x=\frac{3}{2t^2+4}$　$y=\frac{t}{t^2+2}$

↓

$2t^2+4=\frac{3}{x}$

$\Leftrightarrow t^2=\frac{3}{2x}-2$

↓　2乗

$y^2=t^2\cdot\left(\frac{1}{t^2+2}\right)^2$

$=\left(\frac{3}{2x}-2\right)\left(\frac{2x}{3}\right)^2$

$=\frac{2x}{3}-\frac{8}{9}x^2$

$\Leftrightarrow x^2-\frac{3}{4}x+\frac{9}{8}y^2=0$

$\Leftrightarrow \frac{\left(x-\frac{3}{8}\right)^2}{\frac{9}{64}}+\frac{y^2}{\frac{1}{8}}=0$

$\sqrt{\frac{9}{64}-\frac{1}{8}}=\frac{1}{8}$ より

焦点は $\left(\frac{1}{2},0\right),\left(\frac{1}{4},0\right)$

長軸 $\frac{3}{4}$，短軸 $\frac{1}{\sqrt{12}}$ //

[IV]

(1) $W(0)=3$

$P_1=P(W(1)=3)$

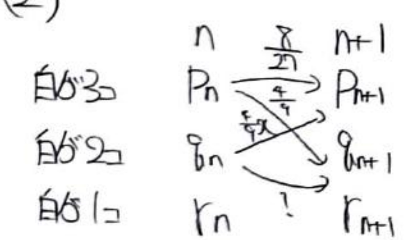
$=\left(\frac{2}{3}\right)^3$

$=\frac{8}{27}$ //

$q_1=P(W(1)=2)$

$={}_3C_2\left(\frac{2}{3}\right)^2\cdot\frac{1}{3}=\frac{4}{9}$ //

(2)



上の遷移図より

$p_{n+1}=\frac{8}{27}p_n+\frac{4x}{9}q_n$ //

上の図の?は

$?=P\begin{pmatrix}①\to①\\②\to②\\③\to③\end{pmatrix}+P\begin{pmatrix}①\to③\\②\to○\\③\to○\end{pmatrix}+P\begin{pmatrix}①\to○\\②\to③\\③\to○\end{pmatrix}$

$=\frac{2}{3}\cdot\frac{2}{3}(1-x)+\frac{1}{3}\cdot\frac{2}{3}x+\frac{2}{3}\cdot\frac{1}{3}x$

$=\frac{4}{9}$

以上より

$q_{n+1}=\frac{4}{9}p_n+\frac{4}{9}q_n$ //

(3)

p_{n+2}

$=\frac{8}{27}p_{n+1}+\frac{4x}{9}q_{n+1}$

$=\frac{8}{27}p_{n+1}+\frac{4x}{9}\left(\frac{4}{9}p_n+\frac{4}{9}q_n\right)$

$=\frac{8}{27}p_{n+1}+\frac{16x}{81}p_n+\frac{4}{9}\left(p_{n+1}-\frac{8}{27}p_n\right)$

$=\frac{20}{27}p_{n+1}+\frac{48x-32}{243}p_n$

$\therefore\ \alpha+\beta=\frac{20}{27}$

$\alpha\beta=\frac{32-48x}{243}$

α,β は

$t^2-\frac{20}{27}t+\frac{32-48x}{243}=0$

の解．

$t=\frac{10}{27}\pm\sqrt{\left(\frac{10}{27}\right)^2-\frac{32-48x}{243}}$

$=\frac{10}{27}\pm\sqrt{\frac{100-96+144x}{27^2}}$

$=\frac{10}{27}\pm\frac{2\sqrt{36x+1}}{27}$

$\therefore\ \alpha=\frac{10-2\sqrt{36x+1}}{27}$

$\beta=\frac{10+2\sqrt{36x+1}}{27}$ //

(4)

$p_0=F(x)+G(x)=1$

$p_1=\alpha F(x)+\beta G(x)=\frac{8}{27}$

↓

$(\alpha-\beta)G(x)=\alpha-\frac{8}{27}$

$\Longleftrightarrow \frac{-4\sqrt{36x+1}}{27}G(x)=\frac{2-2\sqrt{36x+1}}{27}$

$\therefore\ G(x)=\frac{\sqrt{36x+1}-1}{2\sqrt{36x+1}}$ //

$(\beta-\alpha)F(x)=\beta-\frac{8}{27}$

$\Longleftrightarrow \frac{4\sqrt{36x+1}}{27}F(x)=\frac{2+2\sqrt{36x+1}}{27}$

$\therefore\ F(x)=\frac{\sqrt{36x+1}+1}{2\sqrt{36x+1}}$ //

$q_0=H(x)+I(x)=0$

$q_1=\alpha H(x)+\beta I(x)=\frac{4}{9}$

↓

$(\beta-\alpha)H(x)=-\frac{4}{9}$

$\therefore\ H(x)=-\frac{3}{\sqrt{36x+1}}$

$I(x)=\frac{3}{\sqrt{36x+1}}$ //

(5)

r_{n+1}

$=\frac{1}{3}r_n+3\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}p_n+\left(2\cdot\frac{2}{3}\cdot\frac{1}{3}(1-x)+\left(\frac{1}{3}\right)^2x\right)q_n$

$=\frac{1}{3}r_n+\frac{2}{9}p_n+\frac{4-3x}{9}q_n$

$\downarrow\times 3^{n+1}$

$3^{n+1}r_{n+1}=3^nr_n+\left(\frac{2}{9}p_n+\frac{4-3x}{9}q_n\right)3^{n+1}$

$=3^nr_n+\underbrace{\left\{\frac{2}{3}F(x)+\frac{4-3x}{3}H(x)\right\}}_{A}(3\alpha)^n+\underbrace{\left\{\frac{2}{3}G(x)+\frac{4-3x}{3}I(x)\right\}}_{B}(3\beta)^n$

$3^n r_n$ （$n \geq 1$のとき）

$$= 3^0 \underset{0}{r_0} + A\sum_{k=0}^{n-1}(3\alpha)^k + B\sum_{k=0}^{n-1}(3\beta)^k$$

$$= A\frac{1-(3\alpha)^n}{1-3\alpha} + B\frac{1-(3\beta)^n}{1-3\beta}$$

r_n

$$= A\frac{\frac{1}{3^n}-\alpha^n}{1-3\alpha} + B\frac{\frac{1}{3^n}-\beta^n}{1-3\beta}$$

$\frac{r_n}{q_n}$

$$= \frac{A\frac{\frac{1}{3^n}-\alpha^n}{1-3\alpha} + B\frac{\frac{1}{3^n}-\beta^n}{1-3\beta}}{I(\alpha)(\beta^n-\alpha^n)}$$

$$= \frac{\frac{A}{1-3\alpha}\left(\frac{1}{(3\beta)^n}-\left(\frac{\alpha}{\beta}\right)^n\right)+\frac{B}{1-3\beta}\left(\frac{1}{(3\beta)^n}-1\right)}{I(\alpha)\left(1-\left(\frac{\alpha}{\beta}\right)^n\right)}$$

ここで

$$\beta > \frac{10+2}{27} = \frac{4}{9} > \frac{1}{3}$$

より

$$\lim_{n\to\infty}\frac{r_n}{q_n}$$

$$= \frac{-\frac{B}{1-3\beta}}{I(\alpha)}$$

$$= -\frac{\frac{2}{3}G(\alpha)+\frac{4-3x}{3}I(\alpha)}{(1-3\beta)I(\alpha)}$$

↓ $\sqrt{36x+1} = X$とおく

$$= -\frac{\frac{2}{3}\cdot\frac{X-1}{6X}+\frac{4-3x}{3}\cdot\frac{3}{X}}{\left(1-\frac{10+2X}{9}\right)\frac{3}{X}}$$

$$= -\frac{\frac{X-1}{9}+\frac{4-3x}{3}}{-\frac{1}{9}-\frac{2X}{9}}$$

$$= -\frac{X-1+12-9x}{-1-2X}$$

$$= \frac{X-9x+11}{1+2X} < 1$$

$$X - 9x + 11 < 1 + 2X$$

$$\Longleftrightarrow \underset{>0}{10-9x} < X$$

$$X^2 > (10-9x)^2$$

$$\Longleftrightarrow 36x+1 > 81x^2-180x+100$$

$$\Longleftrightarrow 0 > 81x^2-216x+99$$

$$\Longleftrightarrow 0 > 9x^2-24x+11$$

$$\Longleftrightarrow \frac{4-\sqrt{5}}{3} < x < \frac{4+\sqrt{5}}{3}$$

↓ $0<x<1$より

$$\frac{4-\sqrt{5}}{3} < x < 1$$

//