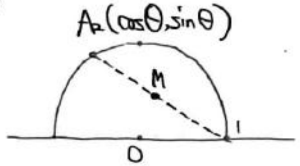
2020 慶應理工

1.

(1)

(i) $A_2(\cos\theta, \sin\theta)$ $(0 \leqq \theta \leqq \pi)$ とする



$M(X, Y)$ とおくと

$$\begin{cases} X = \frac{1}{2}(1+\cos\theta) \Leftrightarrow \cos\theta = 2X-1 \\ Y = \frac{1}{2}\sin\theta \Leftrightarrow \sin\theta = 2Y \end{cases}$$

$\downarrow \cos^2\theta + \sin^2\theta = 1$

$(2X-1)^2 + 4Y^2 = 1$

$\Leftrightarrow \left(X - \frac{1}{2}\right)^2 + Y^2 = \frac{1}{4}$

Mがえがく曲線と実軸の面積は

半円部分より

$\frac{1}{2} \cdot \frac{1}{2}\pi \cdot \frac{1}{2} = \frac{\pi}{8}$

(ii) $z_1 = \cos\varphi + i\sin\varphi$ $(0 \leqq \varphi \leqq \pi)$

とおくと

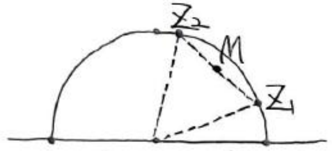
$|z_1|^2 = z_1\overline{z_1} = 1$

$\Leftrightarrow \overline{z_1} = \frac{1}{z_1}$

$z_2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\frac{1}{\overline{z_1}}$

$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)z_1$

$= \cos\left(\varphi + \frac{\pi}{3}\right) + i\sin\left(\varphi + \frac{\pi}{3}\right)$



Mは半径が $\frac{\sqrt{3}}{2}$ の半円周上の

偏角が $30°$ から $150°$ まで動く.

曲線の長さは

$\frac{\sqrt{3}}{2} \cdot 2 \cdot \pi \cdot \frac{1}{3} = \frac{\sqrt{3}}{3}\pi$

(2) C_1 の接点を (s, s^2) とおくと

接線は

$y = 2s(x-s) + s^2$

$= 2sx - s^2$

$\downarrow$ P通る

$t^2 - 4 = 2st - s^2$

$\Leftrightarrow s^2 - 2ts + t^2 - 4 = 0$

$\Leftrightarrow s = t \pm 2$

Aのx座標は $t-2$

$\overrightarrow{OQ} = \frac{2\overrightarrow{OP} + \overrightarrow{OA}}{1+2} = \frac{2}{3}\overrightarrow{OP} + \frac{1}{3}\overrightarrow{OA}$

$\overrightarrow{OR} = \frac{3\overrightarrow{OQ} + 2\overrightarrow{OB}}{2+3}$

$= \frac{3}{5}\left(\frac{2}{3}\overrightarrow{OP} + \frac{1}{3}\overrightarrow{OA}\right) + \frac{2}{5}\overrightarrow{OB}$

$= \frac{2}{5}\overrightarrow{OP} + \frac{1}{5}\overrightarrow{OA} + \frac{2}{5}\overrightarrow{OB}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

$= -\frac{3}{5}\overrightarrow{OP} + \frac{1}{5}\overrightarrow{OA} + \frac{2}{5}\overrightarrow{OB}$

$= -\frac{3}{5}\overrightarrow{OP} + \frac{1}{5}(\overrightarrow{OP} + \overrightarrow{PA})$

$+ \frac{2}{5}(\overrightarrow{OP} + \overrightarrow{PB})$

$= \frac{1}{5}\overrightarrow{PA} + \frac{2}{5}\overrightarrow{PB}$

$\overrightarrow{OR} = \frac{2}{5}\begin{pmatrix} t \\ t^2-4 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} t-2 \\ (t-2)^2 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} t+2 \\ (t+2)^2 \end{pmatrix}$

$= \begin{pmatrix} t + \frac{2}{5} \\ t^2 + \frac{4}{5}t + \frac{4}{5} \end{pmatrix}$

$R(X, Y)$ とおくと

$$\begin{cases} X = t + \frac{2}{5} \Leftrightarrow t = X - \frac{2}{5} \\ Y = t^2 + \frac{4}{5}t + \frac{4}{5} \end{cases}$$

$\therefore Y = \left(X - \frac{2}{5}\right)^2 + \frac{4}{5}\left(X - \frac{2}{5}\right) + \frac{4}{5}$

$= X^2 + \frac{16}{25}$

Rの軌跡の方程式は

$y = x^2 + \frac{16}{25}$

2.

(1) $x = \alpha$ が方程式 $P(x) = 0$ の

2重解ならば, $x = \alpha$ が方程式

$P'(x) = 0$ の解となることを

示せばよい.

$x = \alpha$ が $P(x) = 0$ の2重解とすると

$P(x) = (x-\alpha)^2 Q(x)$ とおける.

両辺 x で微分すると

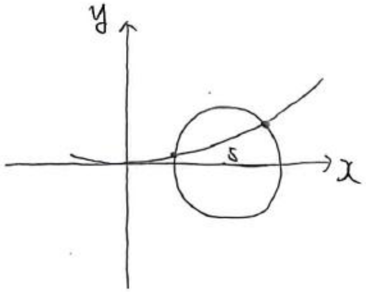
$P'(x) = 2(x-\alpha)Q(x)$

$+ (x-\alpha)^2 Q'(x)$

$= (x-\alpha)\{2Q(x) + (x-\alpha)Q'(x)\}$

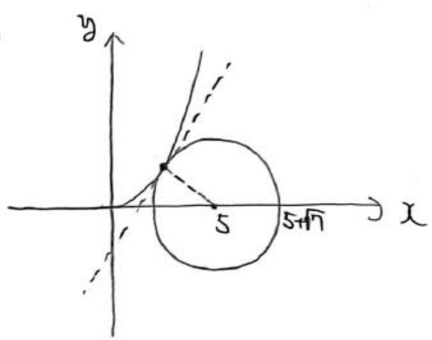
つまり $P'(x) = 0$ は $x = \alpha$ を解にもつ

(2)



$y=kx^2$ で k を限りなく0に近づけるとyも0に近づくので交点が最大で2コもつことは明らか。

(3)



接点を $P(5+\sqrt{7}\cos\theta,\ \sqrt{7}\sin\theta)$ とおく.

傾きに関して

$$2k(5+\sqrt{7}\cos\theta)=\tan\left(\theta-\frac{\pi}{2}\right)=-\frac{1}{\tan\theta}$$

またy座標に関して

$$\sqrt{7}\sin\theta=k(5+\sqrt{7}\cos\theta)^2$$

以上より

$$2k(5+\sqrt{7}\cos\theta)^2=-\frac{5+\sqrt{7}\cos\theta}{\tan\theta}=2\sqrt{7}\sin\theta$$

$$\Leftrightarrow -5-\sqrt{7}\cos\theta=2\sqrt{7}\frac{\sin^2\theta}{\cos\theta}$$

$$\Leftrightarrow -5\cos\theta-\sqrt{7}\cos^2\theta=2\sqrt{7}(1-\cos^2\theta)$$

$$\Leftrightarrow \sqrt{7}\cos^2\theta-5\cos\theta-2\sqrt{7}=0$$

$$\Leftrightarrow (\sqrt{7}\cos\theta+2)(\cos\theta-\sqrt{7})=0$$

$$\therefore \cos\theta=-\frac{2}{\sqrt{7}}$$

$$\sin\theta=\pm\frac{\sqrt{3}}{\sqrt{7}}$$

接点のx座標 $5+\sqrt{7}\cos\theta=3$

y座標 $\pm\sqrt{3}$.

つまり

$$\pm\sqrt{3}=k\cdot3^2$$

$$\therefore k=\pm\frac{\sqrt{3}}{9}$$

3.

(1) $p=\frac{1}{2}$ である.

P(2回ですべて赤)

$$=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{6}\cdot\frac{3}{6}+{}_2C_1\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{6}\cdot\frac{{}_3C_2}{{}_6C_2}+\frac{1}{2}\cdot\frac{1}{2}\left(\frac{{}_3C_2}{{}_6C_2}\right)^2$$

$$=\frac{1}{4}\left(\frac{1}{4}+\frac{3}{15}+\left(\frac{3}{15}\right)^2\right)$$

$$=\frac{1}{4}\cdot\frac{25+20+4}{100}=\frac{49}{400}$$

$P_{3回で5コ}$(すべて赤)

$$=\frac{P(\text{3回で5コのすべて赤})}{P(\text{3回で5コ})}$$

$$=\frac{{}_3C_1\frac{1}{2}\left(\frac{1}{2}\right)^2\cdot\frac{3}{6}\left(\frac{{}_3C_2}{{}_6C_2}\right)^2}{{}_3C_1\frac{1}{2}\left(\frac{1}{2}\right)^2}$$

表が1回

$$=\frac{1}{2}\left(\frac{1}{5}\right)^2=\frac{1}{50}$$

(2) $p=\frac{1}{2}$ で非復元.

P(2回ですべて赤)

$$=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{6}\cdot\frac{2}{5}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{6}\cdot\frac{{}_2C_2}{{}_5C_2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{{}_3C_2}{{}_6C_2}\cdot\frac{1}{4}$$

$$=\frac{1}{20}+\frac{1}{80}+\frac{1}{80}$$

$$=\frac{6}{80}=\frac{3}{40}$$

(3) 非復元.

P(3回で赤,白を2コずつ)

$$=P(\text{表2回, 裏1回})$$

$$={}_3C_1p^2(1-p)\frac{{}_3C_2\cdot{}_3C_2\cdot4!}{{}_6P_4}$$

全体で考える

$$=3p^2(1-p)\frac{3}{5}$$

$$=\frac{9}{5}p^2(1-p)$$

P(3回で赤と白が同じコスウ)

$$=\frac{9}{5}p^2(1-p)+\underbrace{P(\text{3回ですべて裏})}_{(1-p)^3}=1-p$$

$\downarrow \div 1-p$

$$\frac{9}{5}p^2+(1-p)^2=1$$

$$\Leftrightarrow \frac{14}{5}p^2-2p=0$$

$$\therefore p=\frac{5}{7}$$

4.

(1) $f(x)=x$ のとき

$g(x)$

$=\int_0^{2x} e^{-t+x}dt$

$=e^x\left[-e^{-t}\right]_0^{2x}$

$=e^x(1-e^{-2x})$

$=\underline{e^x-e^{-x}}$ //

(2)

$g(-x)$

$=\int_0^{-2x} e^{-f(t+x)}dt$ $\quad t+x=u-x \Leftrightarrow u=t+2x,\ dt=du$

$=\int_{2x}^{0} e^{-f(u-x)}du$

$=-\int_0^{2x} e^{-f(u-x)}du$

$=-g(x)$ $g(x)$は奇関数である.

(3)

$f(x)=\sin x$ のとき

$g(x)=\int_0^{2x} e^{-\sin(t-x)}dt$

$\downarrow t-x=u$

$=\int_{-x}^{x} \underbrace{e^{-\sin u}}_{h(u)}du$

$=H(x)-H(-x)$

$g'(x)$

$=h(x)+h(-x)$

$=\underline{e^{-\sin x}+e^{\sin x}}$ //

(4)

$x^3+3x=\int_0^{2x} e^{-f(t-x)}dt$ $\quad t-x=u,\ dt=du$

$=\int_{-x}^{x} \underbrace{e^{-f(u)}}_{i(u)}du$

$=I(x)-I(-x)$

$\downarrow$ 微分

$3x^2+3=i(x)+i(-x)$

$=e^{-f(x)}+e^{-f(-x)}$

$=2e^{-f(x)}$ $\left(\because f(x)\text{は偶関数}\right)$

$\Leftrightarrow e^{-f(x)}=\frac{3}{2}x^2+\frac{3}{2}$

$\Leftrightarrow -f(x)=\log\left(\frac{3}{2}x^2+\frac{3}{2}\right)$

$\therefore \underline{f(x)=-\log\frac{3}{2}(x^2+1)}$ //

$\int_0^1 f(x)dx$

$=\int_0^1\left\{-\log\frac{3}{2}(x^2+1)\right\}dx$

$=\left[x\left\{-\log\frac{3}{2}(x^2+1)\right\}\right]_0^1 + \int_0^1 x\cdot\frac{\frac{3}{2}\cdot 2x}{\frac{3}{2}(x^2+1)}dx$

$=-\log 3+2\int_0^1 \frac{x^2}{x^2+1}dx$

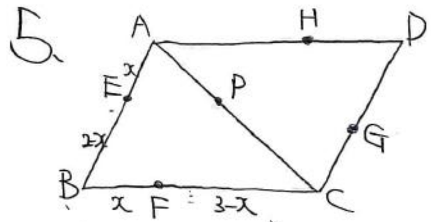
$=-\log 3+2\int_0^1\left(1-\frac{1}{x^2+1}\right)dx$

$=-\log 3+2\left[x-\tan^{-1}x\right]_0^1$

$=-\log 3+2\left(1-\frac{\pi}{4}\right)$

$=\underline{2-\frac{\pi}{2}-\log 3}$ //

5.



$T_1=S\times\frac{1}{2}\times\frac{x}{2}\times\frac{x^2}{4}$

$\therefore \frac{T_1}{S}=\underline{\frac{x^3}{16}}$ //

(2)

$\triangle EBF=\frac{S}{2}\times\frac{2-x}{2}\times\frac{x}{3}=\frac{(2-x)x}{12}S$

$\triangle FCP=\frac{S}{2}\times\frac{3-x}{3}\times\frac{4-x^2}{4}=\frac{(3-x)(4-x^2)}{24}S$

$\triangle EFP=T_2$

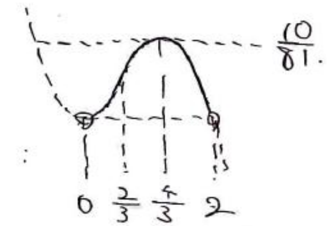
$=\frac{S}{2}-\frac{x^3}{16}S-\frac{(2-x)x}{12}S-\frac{(3-x)(4-x^2)}{24}S$

$=\frac{24-3x^3-4x(2-x)-(6-2x)(4-x^2)}{48}S$

$=\frac{-5x^3+10x^2}{48}S$

$=\frac{S}{48}\underbrace{(-x^3+2x^2)}_{f(x)}S$

$f'(x)=-3x^2+4x=-3x\left(x-\frac{4}{3}\right)$



$x=\frac{4}{3}$ のとき最大値 $\frac{10}{81}$ をとる. //

(3)

$\triangle GHP=T_3$

$=\frac{S}{2}\left(1-\frac{x^2}{4}\cdot\frac{3-x}{3}-\frac{4-x^2}{4}\cdot\frac{x}{2}-\frac{2-x}{3}\cdot\frac{x}{4}\right)$

$=\cdots$

$=\frac{1}{48}(5x^3-2x^2-20x+24)S$

$$\frac{T_3}{S}=\frac{1}{3}$$

$$\Leftrightarrow 5x^3-2x^2-20x+24=16$$

$$\Leftrightarrow (x+2)(x-2)(5x-2)=0$$

$$\therefore \underline{x=\frac{2}{5}}_{/\!/} \quad (\because 0<x<2)$$

$$\frac{3-x}{3}\cdot\frac{2-x}{2}=\frac{x^2}{4}$$

$$\Leftrightarrow 2(3-x)(2-x)=3x^2$$

$$\Leftrightarrow 0=x^2+10x-12$$

$$\therefore \underline{x=-5+\sqrt{37}}_{/\!/}$$

$$(\because 0<x<2)$$

(4)

Pは線分PH上

$\downarrow$

$$\overrightarrow{AP}=(1-t)\overrightarrow{AE}+t\overrightarrow{AH}$$

$$=\frac{(1-t)x}{2}\overrightarrow{AB}+\frac{t(3-x)}{3}\overrightarrow{AD}$$

また

$$\overrightarrow{AP}=\frac{x^2}{4}\overrightarrow{AC}$$

$$=\frac{x^2}{4}(\overrightarrow{AB}+\overrightarrow{AD})$$

より

$$\begin{cases}\dfrac{(1-t)x}{2}=\dfrac{x^2}{4}\\[2mm] \dfrac{t(3-x)}{3}=\dfrac{x^2}{4}\end{cases}$$

$$1-t=\frac{x}{2}$$

$$\Leftrightarrow 1-\frac{x}{2}=t$$