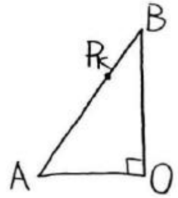
2019 横浜国立大(理工·都市科) 2.

1.

(1)



$\overrightarrow{OP_k}=\left(1-\frac{k}{n}\right)\overrightarrow{OA}+\frac{k}{n}\overrightarrow{OB}$

$|\overrightarrow{OP_k}|^2=\left(1-\frac{k}{n}\right)^2\underbrace{|\overrightarrow{OA}|^2}_{1}+\frac{k^2}{n^2}\underbrace{|\overrightarrow{OB}|^2}_{3}$

$=\frac{4k^2}{n^2}-\frac{2k}{n}+1$

$\therefore |\overrightarrow{OP_k}|=\sqrt{\frac{4k^2}{n^2}-\frac{2k}{n}+1}$

(2)

$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(OP_k)^2}$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{4\frac{k^2}{n^2}-2\frac{k}{n}+1}$

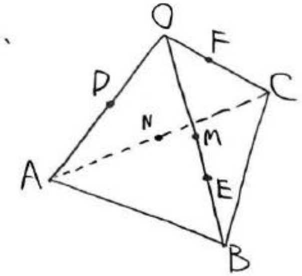
$=\int_0^1\frac{1}{4x^2-2x+1}dx$

$=\int_0^1\frac{1}{4\left(x-\frac{1}{4}\right)^2+\frac{3}{4}}dx$ $\quad x-\frac{1}{4}=\frac{\sqrt{3}}{4}\tan\theta,\ dx=\frac{\sqrt{3}}{4}\frac{1}{\cos^2\theta}d\theta$

$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\frac{3}{4}(\tan^2\theta+1)}\frac{1}{\cos^2\theta}\frac{\sqrt{3}}{4}d\theta$

$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{4}{3}\frac{\sqrt{3}}{4}d\theta=\frac{\sqrt{3}}{6}\pi$

2.



(1)

$|\overrightarrow{MN}|^2$

$=|\overrightarrow{ON}-\overrightarrow{OM}|^2$

$=\left|\frac{\vec{a}+\vec{c}}{2}-\frac{\vec{b}}{2}\right|^2$

$=\frac{1}{4}|\vec{a}-\vec{b}+\vec{c}|^2$

$=\frac{1}{4}(1+1+1-2\underbrace{\vec{a}\cdot\vec{b}}_{\frac{1}{2}}-2\underbrace{\vec{b}\cdot\vec{c}}_{\frac{1}{2}}+2\underbrace{\vec{a}\cdot\vec{c}}_{\frac{1}{2}})$

$=\frac{1}{2}$

$\therefore |\overrightarrow{MN}|=\frac{1}{\sqrt{2}}$

(2)

Pは直線MN上にあるので

$\overrightarrow{OP}=(1-k)\overrightarrow{OM}+k\overrightarrow{ON}$

$=\frac{1-k}{2}\vec{b}+\frac{k}{2}\vec{a}+\frac{k}{2}\vec{c}$

Pは平面DEF上にあるので

$\overrightarrow{OP}=(1-s-t)\overrightarrow{OD}+s\overrightarrow{OE}+t\overrightarrow{OF}$

$=\frac{1-s-t}{2}\vec{a}+\frac{2s}{3}\vec{b}+\frac{2t}{5}\vec{c}$

以上より

$\begin{cases}1-s-t=k\\ \frac{2s}{3}=\frac{1-k}{2}\iff s=\frac{3}{4}(1-k)\\ \frac{2t}{5}=\frac{k}{2}\iff t=\frac{5}{4}k\end{cases}$

$\downarrow$

$1-\frac{3}{4}(1-k)-\frac{5}{4}k=k$

$\iff \frac{1}{4}-\frac{k}{2}=k$

$\therefore k=\frac{1}{6}$

$\therefore \overrightarrow{OP}=\frac{1}{12}\vec{a}+\frac{5}{12}\vec{b}+\frac{1}{12}\vec{c}$

(3) $\overrightarrow{OP}=\frac{5}{6}\overrightarrow{OM}+\frac{1}{6}\overrightarrow{ON}$

PはMNを1:5に内分.

$\therefore |\overrightarrow{MP}|=\frac{1}{6}|\overrightarrow{MN}|=\frac{\sqrt{2}}{12}$

3.

(1) $P(X_n=2)$

$=P$(最小値が2)

$=P$(すべて2以上)$-P$(すべて3以上)

$=\left(\frac{5}{6}\right)^n-\left(\frac{4}{6}\right)^n=\frac{5^n-4^n}{6^n}$

(2) $P(X_2=6)$

$=P$(少なくとも2つ6がある)

$=1-P$(1つだけ6)$-P$(すべて5以下)

$=1-{}_nC_1\frac{1}{6}\left(\frac{5}{6}\right)^{n-1}-\left(\frac{5}{6}\right)^n$

$=\frac{6^n-(n+5)5^{n-1}}{6^n}$

(3)

$P(X_2=6\cap X_n=2)$

$=P(X_n=2)-P(\overline{X_2=6}\cap X_n=2)$

$=P(X_n=2)$

$-P$(1つだけ$6\cap X_n=2$)

$-P$(すべて5以下$\cap X_n=2$)

$=P(X_n=2)$

$-\{P$(1つだけ6∩すべて2以上)

$-P$(1つだけ6∩すべて3以上)$\}$

$-\{P$(すべて5以下∩すべて2以上)

$-P$(すべて5以下∩すべて3以上)$\}$

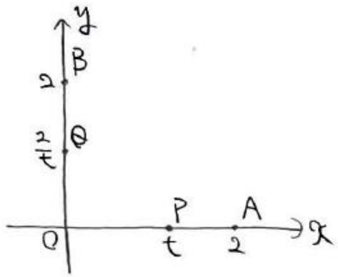
$=\frac{5^n-4^n}{6^n}$

$-{}_nC_1\frac{1}{6}\left(\frac{4}{6}\right)^{n-1}+{}_nC_1\frac{1}{6}\left(\frac{3}{6}\right)^{n-1}$

$-\left(\frac{4}{6}\right)^n+\left(\frac{3}{6}\right)^n$

$=\frac{5^n-(n+8)4^{n-1}+(n+3)3^{n-1}}{6^n}$

4.



(1) $\triangle OPQ = 1$ より

$OQ = \frac{2}{t}$

条件は

$\begin{cases} 0 < t \leqq 2 \\ 0 < \frac{2}{t} \leqq 2 \end{cases}$

$\therefore\ \underline{1 \leqq t \leqq 2}$ //

(2)

線分PQ: $y = -\frac{\frac{2}{t}}{t}x + \frac{2}{t}$

$= -\frac{2}{t^2}x + \frac{2}{t}$

$(0 \leqq x \leqq t)$

$\iff yt^2 - 2t + 2x = 0 \cdots ①$

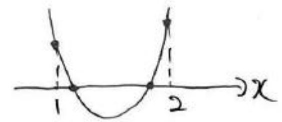
(i) $y=0$ のとき $x=t$

$\therefore\ 1 \leqq x \leqq 2\ (y=0)$

(ii) $y>0$ のとき

①が $1 \leqq t \leqq 2$ で少なくとも1つ実数解をもてばよい。

$2x$ のとき



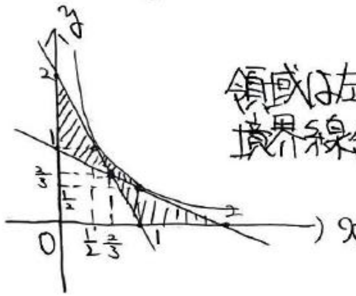
$\begin{cases} y-2+2x \geqq 0 \iff y \geqq -2x+2 \\ 4y-4+2x \geqq 0 \iff y \geqq \frac{1}{2}x+1 \\ 1 \leqq \frac{1}{y} \leqq 2 \iff \frac{1}{2} \leqq y \leqq 1 \\ \frac{D}{4} = 1-2xy \geqq 0 \iff xy \leqq \frac{1}{2} \end{cases}$

1つもつとき

$(y-2+2x)(4y-4+2x) \leqq 0$

$\iff \begin{cases} y \geqq -2x+2 \\ y \leqq -\frac{1}{2}x+1 \end{cases}$

または $\begin{cases} y \leqq -2x+2 \\ y \geqq -\frac{1}{2}x+1 \end{cases}$



領域は左図.
境界線含む.

(3)

S

$= 1 \times \frac{2}{3} \times \frac{1}{2} \times 2$

$+ \int_{\frac{1}{2}}^{\frac{2}{3}} \left(\frac{1}{2x} + 2x - 2\right)dx$

$+ \int_{\frac{2}{3}}^{1} \left(\frac{1}{2x} + \frac{1}{2}x - 1\right)dx$

$= \frac{2}{3} + \left[\frac{1}{2}\log|x| + x^2 - 2x\right]_{\frac{1}{2}}^{\frac{2}{3}}$

$+ \left[\frac{1}{2}\log|x| + \frac{x^2}{4} - x\right]_{\frac{2}{3}}^{1}$

$= \cdots\cdots = \underline{\frac{1}{3} + \frac{1}{2}\log 2}$ //

5.

(1) $f'(x)$

$= -e^{-x}\sin x + e^{-x}\cos x$

$= -e^{-x}(\sin x - \cos x)$

$= -\sqrt{2}e^{-x}\sin\left(x - \frac{\pi}{4}\right)$

x	$0 \cdots \frac{\pi}{4} \cdots \frac{5}{4}\pi \cdots 2\pi$
$f'(x)$	$+\ 0\ -\ 0\ +$
$f(x)$	$0 \nearrow\ \ \searrow\ \ \nearrow$

$f\left(\frac{5}{4}\pi\right) = -\frac{\sqrt{2}}{2}e^{-\frac{5}{4}\pi}$

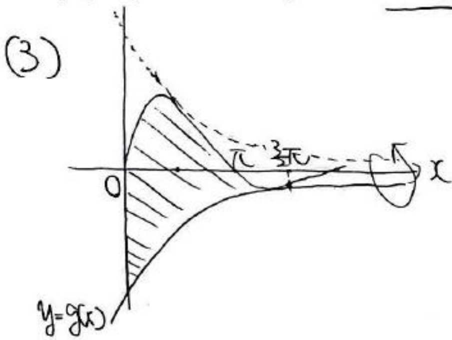
$\underline{x = \frac{5}{4}\pi}$ のとき最小. //

(2) $f(x) = g(x)$

$\iff e^{-x}\sin x = -e^{-x}$

$\iff \sin x = -1$ $\quad \therefore\ \underline{x = \frac{3}{2}\pi}$ //

(3)



V

$= \int_0^{\frac{3}{2}\pi} \{g(x)\}^2 \pi\, dx$

$- \int_{\pi}^{\frac{3}{2}\pi} e^{-2x}\underbrace{\sin^2 x}_{\frac{1-\cos 2x}{2}} \cdot \pi\, dx$

$= \pi\int_0^{\frac{3}{2}\pi} e^{-2x}dx$

$- \frac{\pi}{2}\int_{\pi}^{\frac{3}{2}\pi} e^{-2x}dx + \frac{\pi}{2}\int_{\pi}^{\frac{3}{2}\pi} e^{-2x}\cos 2x\, dx$

ここで

$I = \int e^{-2x}\cos 2x\, dx$

$= -\frac{1}{2}e^{-2x}\cos 2x - \int e^{-2x}\sin 2x\, dx$

$= \frac{1}{2}e^{-2x}(\sin 2x - \cos 2x) - I$

$\therefore\ I = \frac{e^{-2x}}{4}(\sin 2x - \cos 2x) + C$

$\therefore\ V$

$= \pi\left[-\frac{1}{2}e^{-2x}\right]_0^{\frac{3}{2}\pi} - \frac{\pi}{2}\left[-\frac{1}{2}e^{-2x}\right]_{\pi}^{\frac{3}{2}\pi}$

$+ \frac{\pi}{2}\left[\frac{e^{-2x}}{4}(\sin 2x - \cos 2x)\right]_{\pi}^{\frac{3}{2}\pi}$

$= \pi\left(\frac{1}{2} - \frac{1}{2}e^{-3\pi}\right) + \frac{\pi}{4}\left(e^{-3\pi} - e^{-2\pi}\right)$

$+ \frac{\pi}{2}\left\{\frac{e^{-3\pi}}{4} + \frac{e^{-2\pi}}{4}\right\}$

$= \frac{\pi}{2} - \frac{\pi}{8}e^{-3\pi} - \frac{\pi}{8}e^{-2\pi}$

$= \underline{\frac{\pi}{8}\left(4 - e^{-3\pi} - e^{-2\pi}\right)}$ //