2019 横浜市立大

(理, 医, データサイエンス)

[I]

(1) $10 = 2\times 5$

$n = a^4 b$ (a, bは素数)と表せる.

$(a,b) = (2,3), (2,5)$,

$\underline{n = 48, 80}$ //

(2)

$a_1 = 1$ および漸化式の形から

$a_n > 0$. 両辺の逆数をとると

$$\frac{1}{a_{n+1}} = 3\frac{1}{a_n} + 2n$$

$\downarrow$ $b_n = \frac{1}{a_n}$ とおく

$$b_{n+1} = 3b_n + 2n$$

$$\Leftrightarrow b_{n+1} + \alpha(n+1) + \beta = 3(b_n + \alpha n + \beta)$$

$$\Leftrightarrow b_{n+1} = 3b_n + 2\alpha n - \alpha + 2\beta$$

$\therefore \alpha = 1,\ \beta = \frac{1}{2}$.

$$b_{n+1} + (n+1) + \frac{1}{2} = 3\left(b_n + n + \frac{1}{2}\right)$$

$\downarrow$

$$b_n + n + \frac{1}{2} = \left(b_1 + 1 + \frac{1}{2}\right)3^{n-1}$$

$$\Leftrightarrow b_n = \frac{5}{2}\cdot 3^{n-1} - n - \frac{1}{2}$$

$$\therefore \underline{a_n = \frac{2}{5\cdot 3^{n-1} - 2n - 1}}$$ //

(3)

$$\frac{1}{\tan\frac{\pi}{24}}$$

$$= \frac{1}{\tan\frac{1}{2}\cdot\frac{\pi}{12}}$$

ここで

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$\Downarrow$

$$\frac{1}{\tan^2\frac{\theta}{2}} = \frac{1+\cos\theta}{1-\cos\theta}$$

$$= \sqrt{\frac{1}{\tan^2\frac{1}{2}\cdot\frac{\pi}{12}}}$$

$$= \sqrt{\frac{1+\cos\frac{\pi}{12}}{1-\cos\frac{\pi}{12}}}$$

$$= \sqrt{\frac{\left(1+\cos\frac{\pi}{12}\right)^2}{\sin^2\frac{\pi}{12}}}$$

$$= \frac{1+\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}$$

$\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

$\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$$= \frac{4+\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$$

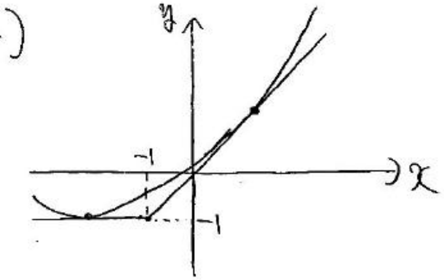
$$= \frac{(4+\sqrt{6}+\sqrt{2})(\sqrt{6}+\sqrt{2})}{4}$$

$$= \sqrt{6}+\sqrt{2}+\frac{8+2\sqrt{12}}{4}$$

$$= \sqrt{6}+\sqrt{2}+2+\sqrt{3}$$

$$\therefore \frac{1}{\tan\frac{\pi}{24}} - \sqrt{2} - \sqrt{3} - \sqrt{6} = \underline{2}$$ //

(4)



$$x^2 + bx + c = x$$

$$\Leftrightarrow x^2 + (b-1)x + c = 0$$

$$D = (b-1)^2 - 4c = 0 \cdots ①$$

また

$$y = \left(x+\frac{b}{2}\right)^2 - \frac{b^2}{4} + c$$

$\downarrow$

$$-\frac{b^2}{4} + c = -1 \cdots ②$$

①, ②より

$$(b-1)^2 = 4\left(\frac{b^2}{4} - 1\right)$$

$$\Leftrightarrow -2b + 1 = -4$$

$$\therefore \underline{b = \frac{5}{2}}$$ //

$\downarrow$

$$\frac{9}{4} - 4c = 0 \quad \therefore \underline{c = \frac{9}{16}}$$ //

[II]

(1)

$$\frac{xyz-1}{(x-1)yz}<\frac{xyz}{(x-1)yz}$$
$$=\frac{x}{x-1}$$
$$<\frac{xyz}{(x-1)(y-1)(z-1)}$$

(2)

$$\frac{xyz-1}{(x-1)(y-1)(z-1)}$$
$$>\frac{xyz-1}{(x-1)yz}$$
$$>\frac{xyz-yz}{(x-1)yz}=1$$

また $x\geqq 3$, $y\geqq 5$, $z\geqq 7$ より

$$\begin{cases}\dfrac{x}{x-1}\leqq\dfrac{3}{2}\\[2mm]\dfrac{y}{y-1}\leqq\dfrac{5}{4}\\[2mm]\dfrac{z}{z-1}\leqq\dfrac{7}{6}\end{cases}$$

$$\therefore \frac{xyz-1}{(x-1)(y-1)(z-1)}$$
$$<\frac{xyz}{(x-1)(y-1)(z-1)}\leqq\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}=\frac{35}{16}<3$$

$$\therefore 1<\frac{xyz-1}{(x-1)(y-1)(z-1)}<3$$

(3) 3つの解を α,β,γ とすると

$$\begin{cases}\alpha+\beta+\gamma=-A\\ \alpha\beta+\beta\gamma+\gamma\alpha=B\\ \alpha\beta\gamma=-C\end{cases}$$

また

$$\frac{A+B+2C+2}{A+B+C+1}$$
$$=1+\frac{C+1}{A+B+C+1}$$
$$=1+\frac{-\alpha\beta\gamma+1}{-\alpha-\beta-\gamma+\alpha\beta+\beta\gamma+\gamma\alpha-\alpha\beta\gamma+1}$$
$$=1+\frac{\alpha\beta\gamma-1}{\alpha\beta\gamma-\alpha\beta-\beta\gamma-\gamma\alpha+\alpha+\beta+\gamma-1}$$
$$=1+\frac{\alpha\beta\gamma-1}{(\alpha-1)(\beta-1)(\gamma-1)}$$

整数より(2)より2である.

$$\frac{\alpha\beta\gamma-1}{(\alpha-1)(\beta-1)(\gamma-1)}=2\cdots①$$

ちなみに $g(1)=1+A+B+C$ となり, これは0にならないので

1は解でないので解は3以上.

$\alpha<\beta<\gamma$ とする.

もし $\alpha\geqq 5$ とすると

$$\frac{\alpha\beta\gamma-1}{(\alpha-1)(\beta-1)(\gamma-1)}$$
$$<\frac{\alpha\beta\gamma}{(\alpha-1)(\beta-1)(\beta-1)}$$
$$\leqq\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{9}{8}=\frac{105}{64}$$

これは①を満たさないので $\alpha=3$

$$\frac{3\beta\gamma-1}{2(\beta-1)(\gamma-1)}=2$$

$\Leftrightarrow 3\beta\gamma-1=4(\beta-1)(\gamma-1)$

$\Leftrightarrow 3\beta\gamma-1=4\beta\gamma-4\beta-4\gamma+4$

$\Leftrightarrow 0=\beta\gamma-4\beta-4\gamma+5$

$\Leftrightarrow 11=(\beta-4)(\gamma-4)$

$\therefore \beta=5$, $\gamma=15$

$\therefore A=-23$

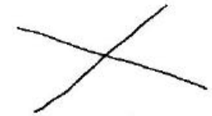
$B=15+75+45=135$

$C=-225$ //

[III]

(1) n本引いたときの面の数を a_nとする.

$a_1=2$



$a_2=4$

$a_3=7$

n本目でn個の面を横切って新たにn個の面を生むので

$$a_n=a_{n-1}+n$$

これより

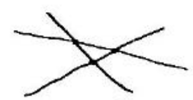
$a_5=a_4+5$

$=a_3+4+5=16$ //

(2) n本引いたときの最大の交点数をb_nとする.

$b_1=0$

$b_2=1$



$b_3=3$

n本目で既にあるn−1本と交わり
それだけ交点が増えるので

$$b_n = b_{n-1} + n - 1$$
$$\downarrow$$
$$b_{n+1} = b_n + n$$

$n \geqq 2$のとき

$$b_n = b_1 + \sum_{k=1}^{n-1} k$$
$$= \frac{1}{2}(n-1)n$$

これは$n=1$のときも成立.

$$\therefore\ \underline{b_n = \frac{1}{2}(n-1)n}\ /\!/$$

(3) (1)より $a_{n+1} = a_n + n + 1$

$n \geqq 2$のとき

$$a_n = a_1 + \sum_{k=1}^{n-1}(k+1)$$
$$= 2 + 2 + 3 + \cdots + n$$
$$= 1 + \frac{1}{2}n(n+1)$$

これは$n=1$のとき成立.

$$\therefore\ \underline{a_n = \frac{n^2+n+2}{2}}\ /\!/$$

[IV]

(1) $\displaystyle\int \frac{1}{\sin x \cos x}dx$

$$= \int \frac{1}{\tan x}\cdot\frac{1}{\cos^2 x}dx \qquad \left(t = \tan x,\ dt = \frac{1}{\cos^2 x}dx\right)$$
$$= \int \frac{1}{t}dt$$
$$= \log|t| + C = \log|\tan x| + C$$

(2) $f(x) = \dfrac{\cos x}{\sin^n x}$ とおく.

$$f'(x) = \frac{-\sin^{n+1}x + n\cos^2 x \cdot \sin^{n-1}x}{\sin^{2n}x}$$
$$= \underline{\frac{-\sin^2 x + n\cos^2 x}{\sin^{n+1}x}}\ /\!/$$

(3)

$$\left(\frac{1}{(n-1)\sin^{n-1}x}\right)'$$
$$= \left(\frac{1}{n-1}(\sin x)^{1-n}\right)'$$
$$= -(\sin x)^{-n}\cos x$$
$$= -\frac{\cos x}{\sin^n x}$$
$$= -\frac{\cos^2 x}{\sin^n x \cos x}$$
$$= \frac{\sin^2 x - 1}{\sin^n x \cos x}$$
$$= \frac{1}{\sin^{n-2}x\cos x} - \frac{1}{\sin^n x \cos x}$$

積分すると

$$\frac{1}{(n-1)\sin^{n-1}x} = \int \frac{1}{\sin^{n-2}x\cos x}dx - \int \frac{1}{\sin^n x \cos x}dx$$

$$\therefore \int \frac{dx}{\sin^n x\cos x} = -\frac{1}{(n-1)\sin^{n-1}x} + \int \frac{dx}{\sin^{n-2}x\cos x}$$

(4)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin^3 x \cos x}$$
$$= \left[-\frac{1}{2\sin^2 x}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin x\cos x}$$
$$= \left[-\frac{1}{2\sin^2 x} + \log|\tan x|\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= -\frac{2}{3} + \log\sqrt{3} - \left(-2 + \log\left|\frac{1}{\sqrt{3}}\right|\right)$$
$$= -\frac{2}{3} + \log\sqrt{3} + \frac{6}{3} + \log\sqrt{3}$$
$$= \underline{\frac{4}{3} + \log 3}\ /\!/$$

[V]

(1)

$\int_{-1}^{3} f(x)dx$

$= \int_{-1}^{0} a(x+1)dx + \int_{0}^{3}(bx+a)dx$

$= \left[\frac{a}{2}(x+1)^2\right]_{-1}^{0} + \left[\frac{b}{2}x^2+ax\right]_{0}^{3}$

$= \frac{a}{2} + \frac{9}{2}b + 3a$

$= \frac{7}{2}a + \frac{9}{2}b = 1 \cdots ①$

$\int_{-1}^{3} xf(x)dx$

$= \int_{-1}^{0}(ax^2+ax)dx + \int_{0}^{3}(bx^2+ax)dx$

$= \left[\frac{a}{3}x^3\right]_{-1}^{0} + \left[\frac{b}{3}x^3\right]_{0}^{3} + \left[\frac{a}{2}x^2\right]_{-1}^{3}$

$= \frac{a}{3} + 9b + 4a$

$= \frac{13}{3}a + 9b = \frac{2}{3}$

$-)\ 7a + 9b = 2 \leftarrow ① \times 2$

$-\frac{8}{3}a = -\frac{4}{3}$

$\therefore \underline{a = \frac{1}{2} \quad b = -\frac{1}{6}}$ //

(2)

$E(X^2)$

$= \int_{-1}^{0} \frac{1}{2}(x^3+x^2)dx + \int_{0}^{3}\left(-\frac{x^3}{6}+\frac{x^2}{2}\right)dx$

$= \left[\frac{1}{8}x^4\right]_{-1}^{0} + \left[-\frac{x^4}{24}\right]_{0}^{3} + \left[\frac{x^3}{6}\right]_{-1}^{3}$

$= -\frac{1}{8} - \frac{27}{8} + \frac{28}{6}$

$= -\frac{7}{2} + \frac{14}{3}$

$= \frac{28-21}{6} = \frac{7}{6}$

$V(X) = E(X^2) - (E(X))^2$

$= \frac{7}{6} - \frac{4}{9}$

$= \frac{21-8}{18}$

$= \underline{\frac{13}{18}}$ //

(3)

(ア)

$E(\bar{Y})$

$= E(Y)$

$= 18E(X) + 5$

$= \underline{17}$ //

$V(\bar{Y})$

$= V\left(\frac{Y_1 + \cdots + Y_{117}}{117}\right)$

$= \frac{1}{117^2}\{V(Y_1) + \cdots + V(Y_{117})\}$

$= \frac{1}{117}V(Y)$

$= \frac{1}{117}V(18X+5)$

$= \frac{1}{117}V(18X)$

$= \frac{324}{117}V(X)$

$= \frac{18}{117} \cdot 13 = \underline{2}$ //

(イ)

$P(16 \leqq \bar{Y} \leqq 18)$

$= P(-1 \leqq \bar{Y} - 17 \leqq 1)$

$= P\left(-\frac{1}{\sqrt{2}} \leqq \frac{\bar{Y}-17}{\sqrt{2}} \leqq \frac{1}{\sqrt{2}}\right)$

$= P\left(-\frac{\sqrt{2}}{2} \leqq \frac{\bar{Y}-17}{\sqrt{2}} \leqq \frac{\sqrt{2}}{2}\right)$

$\sqrt{2} \fallingdotseq 1.4142$ より

$0.7071 < \frac{\sqrt{2}}{2} < 0.7072$

$P\left(\left|\frac{\bar{Y}-17}{\sqrt{2}}\right| \leqq 0.70\right) < P\left(\left|\frac{\bar{Y}-17}{\sqrt{2}}\right| \leqq \frac{\sqrt{2}}{2}\right)$

$< P\left(\left|\frac{\bar{Y}-17}{\sqrt{2}}\right| \leqq 0.71\right)$

$\Leftrightarrow 0.2580 \times 2 < P\left(\left|\frac{\bar{Y}-17}{\sqrt{2}}\right| \leqq \frac{\sqrt{2}}{2}\right)$

$< 0.2611 \times 2$

$\Leftrightarrow 0.516 < P\left(\left|\frac{\bar{Y}-17}{\sqrt{2}}\right| \leqq \frac{\sqrt{2}}{2}\right)$

< 0.5222

求める確率の近似値は

$\underline{0.52}$ //