2019 慶應 理工

1.

(1)

(i) $f(x)=\sqrt{x}\log x$ とおく

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}\log x+x^{-\frac{1}{2}}$

$=\frac{1}{2\sqrt{x}}(\log x+2)$

x	$0\cdots\frac{1}{e^2}\cdots$
$f'(x)$	$-\ 0\ +$
$f(x)$	$\searrow -\frac{2}{e} \nearrow$

以上より $f(x)\geqq-\frac{2}{e}>-1$

(ii)

(i)より

$x\log x>-\sqrt{x}$

$\therefore -\sqrt{x}<x\log x<0$

$\lim_{x\to+0}(-\sqrt{x})=0$ より

$\lim_{x\to+0}x\log x=0$

(2)

$|z^2-1|=|z+1||z-1|=\frac{1}{2}$

$\therefore |z+1|=\frac{1}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{5}+\sqrt{3}}{2}$

$\downarrow z=x+yi$

$|z-1|^2=(x-1)^2+y^2=\frac{8-2\sqrt{15}}{4}$

$-)\ |z+1|^2=(x+1)^2+y^2=\frac{8+2\sqrt{15}}{4}$

$-4x=-\sqrt{15}$

$\therefore x=\frac{\sqrt{15}}{4}\quad x+1=\frac{\sqrt{15}+4}{4}$

$\downarrow$

$\frac{31+8\sqrt{15}}{16}+y^2=\frac{32+8\sqrt{15}}{16}$

$\therefore y^2=\frac{1}{16}\quad y=\pm\frac{1}{4}$

$\therefore z=\frac{\sqrt{15}\pm i}{4}$

(3)

$f(x)$

$=|x\vec{a}-\vec{p}|^2$

$=(x-\cos\theta)^2+(x-\sin\theta)^2$ とおく

$f'(x)$

$=2(x-\cos\theta)+2(x-\sin\theta)$

$=4x-2\sin\theta-2\cos\theta$

$\frac{(x\vec{a}-\vec{p})\cdot\vec{a}}{|x\vec{a}-\vec{p}|^2}=\frac{2x-\cos\theta-\sin\theta}{(x-\cos\theta)^2+(x-\sin\theta)^2}$

$=\frac{f'(x)}{2f(x)}$

$(与式)=\frac{1}{2}\left[\log f(x)\right]_0^{\frac{\sqrt{2}}{4}}$

$=\frac{1}{2}\log\left\{\left(\frac{\sqrt{2}}{4}-\cos\theta\right)^2+\left(\frac{\sqrt{2}}{4}-\sin\theta\right)^2\right\}$

$=\frac{1}{2}\log\left(\frac{5}{4}-\frac{\sqrt{2}}{2}\sin\theta-\frac{\sqrt{2}}{2}\cos\theta\right)$

$=\frac{1}{2}\log\left\{\frac{5}{4}-\sin\left(\theta+\frac{\pi}{4}\right)\right\}$

最大値

$\frac{1}{2}\log\frac{9}{4}=\log\frac{3}{2}$

2.

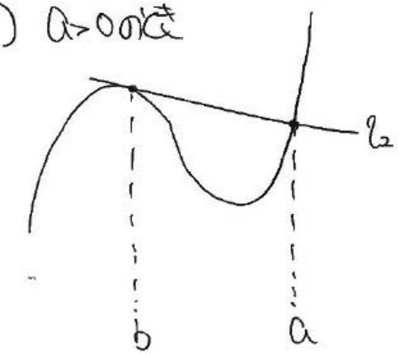
$f(x)=\begin{cases}2x^2-4x & (x\geqq0)\\ -\frac{9}{8}x^2-4x & (x<0)\end{cases}$

$f'(x)=\begin{cases}4x-4 & (x\geqq0)\\ -\frac{9}{4}x-4 & (x<0)\end{cases}$

x	$\cdots$	$-\frac{16}{9}$	$\cdots$	0	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$+$
$f(x)$	$\nearrow$	$\frac{32}{9}$	$\searrow$		$\searrow$	-2	$\nearrow$

$(極値の和)=\frac{32}{9}-2=\frac{14}{9}$

(2) $a>0$ のとき



$l_2: y=\left(-\frac{9}{4}b-4\right)(x-b)-\frac{9}{8}b^2-4b$

$=\left(-\frac{9}{4}b-4\right)x+\frac{9}{8}b^2$

$\downarrow (a, f(a))$ 通る

$2a^2-4a=\left(-\frac{9}{4}b-4\right)a+\frac{9}{8}b^2$

$\Leftrightarrow 2a^2=-\frac{9}{4}ab+\frac{9}{8}b^2$

$\Leftrightarrow 16a^2+18ab-9b^2=0$

$\Leftrightarrow (8a-3b)(2a+3b)=0$

$b=-\frac{2}{3}a$ (明らかに $b<0$)

$a<0$ のとき

$l_2: y=(4b-4)(x-b)+2b^2-4b$

$=(4b-4)x-2b^2$

$\downarrow (a, f(a))$ 通る

$-\frac{9}{8}a^2-4a=(4b-4)a-2b^2$

$\Leftrightarrow -\frac{9}{8}a^2 = 4ba - 2b^2$

$\Leftrightarrow 16b^2 - 32ba - 9a^2 = 0$

$\Leftrightarrow (4b+a)(4b-9a) = 0$

$\therefore \underline{b = -\frac{1}{4}a}$

(3)

$a>0$のとき

(ℓ₁の傾き)・(ℓ₂の傾き)

$= (4a-4)(-\frac{9}{4}b-4)$　$b=-\frac{2}{3}a$

$= -9ab - 16a + 9b + 16$

$= 6a^2 - 22a + 16 = -1$

$\Leftrightarrow 6a^2 - 22a + 17 = 0$

$\therefore a = \frac{11 \pm \sqrt{19}}{6}$

$a<0$のとき

$(-\frac{9}{4}a-4)(4b-4)$

$= -9ab + 9a - 16b + 16$　$b=-\frac{a}{4}$

$= \frac{9}{4}a^2 + 13a + 16 = -1$

$\Leftrightarrow 9a^2 + 52a + 68 = 0$

$\Leftrightarrow (9a+34)(a+2) = 0$

$\therefore a = -\frac{34}{9}, -2$

これらの中で最大は$\frac{11+\sqrt{19}}{6}$，最小は$-\frac{34}{9}$

3.

(1)

P(n=6で3枚目で終わり)

$= \frac{4\cdot 3!}{{}_6P_3} = \underline{\frac{1}{5}}$ …ケ

(1 2 3)
(2 3 4)
(3 4 5)
(4 5 6)

P(n=6で5枚目で終わり)

= P(n=6で4枚目で終わらない)

(1 2 4 5)　(2 3 5 6)
(1 2 4 6)
(1 2 5 6)
(1 3 4 6)
(1 3 5 6)

= P(最初の4回で上の組が出る)

$= \frac{6\cdot 4!\cdot 2}{{}_6P_5} = \underline{\frac{2}{5}}$ …コ

P(n=6で5枚目で終わり(5枚目が2))

$= \frac{2\cdot 4!\cdot 1}{6\cdot 4!\cdot 2} = \underline{\frac{1}{6}}$ …サ

(2)

P(4枚目で終わりかつ4枚が連続)

$= \frac{(n-3)\cdot 3!\cdot 2}{{}_nP_4}$

(1,…,4)
∫
(n-3,…,n)

例えば(1,…,4)なら
(1,2,4)→3
(1,3,4)→2

$= \frac{(n-3)\cdot 12}{n(n-1)(n-2)(n-3)}$

$= \underline{\frac{12}{n(n-1)(n-2)}}$ …シ

P(4枚目で終わりかつ4枚が連続しない)

(…) ……… n-4とおり
(…) ……… n-5とおり
(…) ……… n-5とおり
⋮
……… (…) n-4とおり
n-4コ

$= \frac{\{2(n-4)+(n-5)(n-4)\}\times 3\times 3!}{{}_nP_4}$

$= \frac{(n-4)\times 18}{n(n-1)(n-2)}$

以上より

P(4枚目で終わり)

$= \frac{12+18(n-4)}{n(n-1)(n-2)} = \underline{\frac{6(3n-10)}{n(n-1)(n-2)}}$

4.

(1) $f(x) = xe^{-bx}$ $(x \geq 0)$

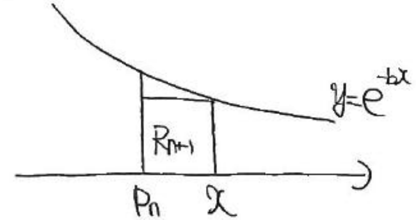
$f'(x) = e^{-bx} - bxe^{-bx}$

$= (1-bx)e^{-bx}$

x	0	$\cdots$	$\frac{1}{b}$	$\cdots$
$f'(x)$		$+$		$-$
$f(x)$	0	↗	$\frac{1}{eb}$	↘

$x = \underline{\frac{1}{b}}$で最大．$\therefore a_1 = \underline{\frac{1}{eb}}$

(2)



$a_{n+1} = (x-p_n)e^{-bx}$

$\frac{da_{n+1}}{dx} = e^{-bx} - b(x-p_n)e^{-bx}$

$= (1+bp_n-bx)e^{-bx}$

x	p_n	$\cdots$	$p_n+\frac{1}{b}$	$\cdots$
$\frac{da_{n+1}}{dx}$		$+$	0	$-$
a_{n+1}		↗		↘

$\therefore p_{n+1} = p_n + \frac{1}{b}$

$P_1 = \frac{1}{b}$ より $P_n = \frac{n}{b}$ //

(3)

$a_n = (P_n - P_{n-1}) e^{-bP_n}$

$= \frac{1}{b} e^{-n}$

$S_n = \sum_{k=1}^{n} \frac{1}{b} e^{-k}$

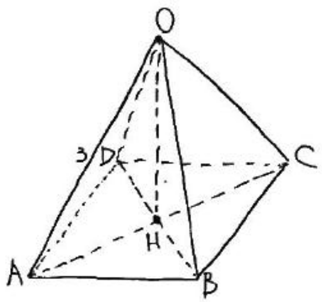
$= \frac{1}{b} \cdot \frac{\frac{1}{e} - e^{-n} e^{-1}}{1 - \frac{1}{e}}$

$= \frac{1 - e^{-n}}{b(e-1)}$ //

$\lim_{n \to \infty} S_n = \frac{1}{b(e-1)} = 1$

$\therefore b = \frac{1}{e-1}$ //

5.



(1)

$AH = 3\sin\theta$

$\therefore AC = 6\sin\theta$ //

(正四角錐O-ABCD)

$= 18\sin^2\theta \times 3\cos\theta \times \frac{1}{3}$

$= 18\sin^2\theta\cos\theta = f(\theta)$ とおく.

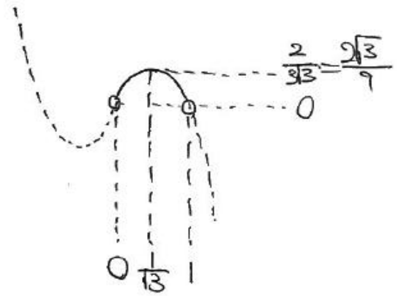
(2)

$f(\theta) = 18\cos\theta - 18\cos^3\theta$ 　$\cos\theta = t$ とおく

$= 18(t - t^3)$ 　$(0 < t < 1)$

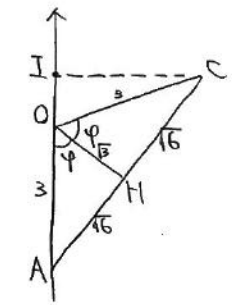
$= 18\, g(t)$ とする.

$g'(t) = 1 - 3t^2$



$\therefore \max f(\theta) = 18 \cdot \frac{2\sqrt{3}}{9} = 4\sqrt{3}$ //

$\cos\varphi = \frac{1}{\sqrt{3}}$ とする



$CI = 3\sin(\pi - 2\varphi)$

$= 3\sin 2\varphi$

$= 6\sin\varphi\cos\varphi$

$= 6 \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = 2\sqrt{2}$

$OI = 3\cos(\pi - 2\varphi)$

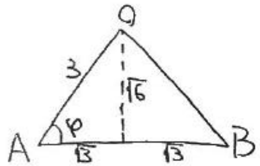
$= -3\underbrace{\cos 2\varphi}_{2\cos^2\varphi - 1}$

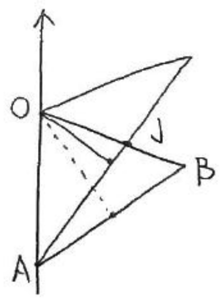
$= 3 - 6\cos^2\varphi$

$= 1$

$K_1 = 2\sqrt{2} \times \sqrt{2} \times \pi \times 4 \times \frac{1}{3} \times \frac{3}{4}$

$= 8\pi$ //





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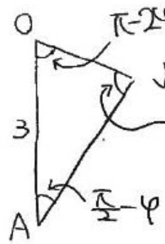

$\pi - (\pi - 2\varphi) - (\frac{\pi}{2} - \varphi)$

$= 3\varphi - \frac{\pi}{2}$

正弦定理

$\frac{3}{\sin(3\varphi - \frac{\pi}{2})} = \frac{OJ}{\sin(\frac{\pi}{2} - \varphi)} = \frac{AJ}{\sin(\pi - 2\varphi)}$

$\therefore OJ = 3 \times \frac{\sin(\frac{\pi}{2} - \varphi)}{\sin(3\varphi - \frac{\pi}{2})}$

$= 3 \times \frac{\cos\varphi}{-\cos 3\varphi}$

$= -3 \times \frac{1}{4\cos^2\varphi - 3}$

$= -3 \times \frac{1}{-\frac{5}{3}} = \frac{9}{5}$

$r = OJ \times \sin(\pi - 2\varphi)$

$= \frac{9}{5}\sin 2\varphi$

$= \frac{18}{5} \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{6\sqrt{2}}{5}$ とする

(K_1とK_2の共通部分)

$= r^2\pi \times 3 \times \frac{1}{3}$

$= \frac{72}{25}\pi$ //