2019 慶應(医)

[I]

(1)

$y = 2\cos^2\theta - \sqrt{3}\cos\theta\sin\theta - \sin^2\theta$

$= 1 + \cos 2\theta - \frac{\sqrt{3}}{2}\sin 2\theta - \frac{1-\cos 2\theta}{2}$

$= \frac{3}{2}\cos 2\theta - \frac{\sqrt{3}}{2}\sin 2\theta + \frac{1}{2}$

$= \sqrt{3}\left(\cos 2\theta \cos\frac{\pi}{6} - \sin 2\theta \sin\frac{\pi}{6}\right) + \frac{1}{2}$

$= \sqrt{3}\cos\left(2\theta + \frac{\pi}{6}\right) + \frac{1}{2}$

$2\theta + \frac{\pi}{6} = 2\pi \Leftrightarrow \theta = \underline{\frac{11}{12}\pi}$

のとき最大値 $\underline{\sqrt{3} + \frac{1}{2}}$

(2)

$P(A \cap B)$

$= P(白\times 2 \to 黒\times 2) + P(黒\times 2 \to 白\times 2)$

$= \frac{1}{{}_6C_2}\cdot\frac{{}_3C_2}{{}_4C_2} + \frac{{}_3C_2}{{}_6C_2}\cdot\frac{1}{{}_4C_2} = \underline{\frac{1}{15}}$

$P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{1 + {}_3C_2}{{}_6C_2} + P(Bが白\times 2) + P(Bが黒\times 2) - \frac{1}{15}$

$= \frac{4}{15} + \frac{{}_4C_2}{{}_6C_2}\cdot\frac{1}{{}_4C_2} + \underbrace{\frac{{}_3C_2}{{}_6C_2}\cdot\frac{{}_3C_2}{{}_4C_2}}_{\boxed{Aが黒なし}} + \underbrace{\frac{3\cdot 3}{{}_6C_2}\cdot\frac{1}{{}_4C_2}}_{\boxed{Aが黒\times 1}} - \frac{1}{15}$

$= \underline{\frac{7}{15}}$

※K君とO君が同時に2つずつ取ることと同じなのでK君とO君は対等. つまり $P(A) = P(B) = \frac{4}{15}$ となる.

(3)

$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$

$\triangle ABC$

$= \frac{1}{2}\sqrt{|\overrightarrow{AB}|^2|\overrightarrow{AC}|^2 - (\overrightarrow{AB}\cdot\overrightarrow{AC})^2}$

$= \frac{1}{2}\sqrt{5\cdot 10 - 1} = \underline{\frac{7}{2}}$

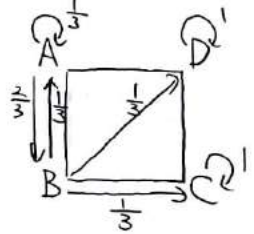
平面ABC: $\frac{x}{1} + \frac{y}{2} + \frac{z}{-3} = 1$

$\Leftrightarrow 6x + 3y - 2z - 6 = 0$

法線ベクトル $\begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}$ 大きさは 7

$\therefore \vec{p} = \underline{\begin{pmatrix} \frac{6}{7} \\ \frac{3}{7} \\ -\frac{2}{7} \end{pmatrix}}$

[II]



(1)

$P_n = \underbrace{P_{n-1}\times\frac{1}{3}}_{\boxed{A\to A}} + \underbrace{P_{n-2}\times\frac{2}{3}\times\frac{1}{3}}_{\boxed{A\to B\to A}}$

$= \underline{\frac{1}{3}}P_{n-1} + \underline{\frac{2}{9}}P_{n-2}$

(2)

(特) $\alpha^2 = \frac{1}{3}\alpha + \frac{2}{9}$

$\Leftrightarrow 9\alpha^2 - 3\alpha - 2 = 0$

$\Leftrightarrow (3\alpha + 1)(3\alpha - 2) = 0$

$\therefore \alpha = -\frac{1}{3}, \frac{2}{3}$

これより

$P_n - \underline{\frac{2}{3}}P_{n-1} = \underline{-\frac{1}{3}}\left(P_{n-1} - \frac{2}{3}P_{n-2}\right)$

(3)

$P_n - \frac{2}{3}P_{n-1} = \left(\underbrace{P_2}_{\frac{1}{9}} - \frac{2}{3}\underbrace{P_1}_{\frac{1}{3}}\right)\left(-\frac{1}{3}\right)^{n-2}$

$= -\frac{1}{9}\left(-\frac{1}{3}\right)^{n-2}$

$= \underline{-\left(-\frac{1}{3}\right)^n}$

(4)

$P_n = き\left(\frac{2}{3}\right)^n + け\left(-\frac{1}{3}\right)^n$

とおける.

$P_1 = \frac{2}{3}き - \frac{1}{3}け = \frac{1}{3} \cdots ①$

$P_2 = \frac{4}{9}き + \frac{1}{9}け = \frac{1}{9}$ ②

$① \times \frac{1}{3} + ②$

$\frac{6}{9}き = \frac{2}{9}$ $\therefore き = \frac{1}{3}$

解くと $け = -\frac{1}{3}$

$\therefore P_n = \frac{1}{3}\left(\frac{2}{3}\right)^n + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^n$

(5)

n回目 $\xrightarrow{\frac{1}{3}}$ n+1回目

A $P_n \xrightarrow{\frac{1}{3}} P_{n+1}$

B $1 - P_n - 2Q_n \nearrow \frac{1}{3}$

$P_{n+1} = \frac{1}{3}P_n + \frac{1}{3}(1 - P_n - 2Q_n)$

$= \frac{1}{3} - \frac{2}{3}Q_n$

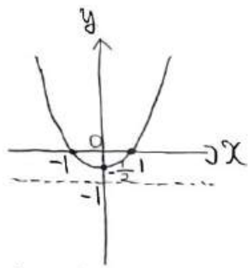
$\Leftrightarrow Q_n = \frac{1}{2} - \frac{3}{2}P_{n+1}$

$= \frac{1}{2} - \frac{1}{2}\left(\frac{2}{3}\right)^{n+1} + \frac{1}{2}\left(-\frac{1}{3}\right)^{n+1}$

$= \underline{\frac{1}{2}} + \underline{\left(-\frac{1}{3}\right)}\underline{\left(\frac{2}{3}\right)^n} + \underline{\left(-\frac{1}{6}\right)}\underline{\left(-\frac{1}{3}\right)^n}$

[III]

(1)



放物線の定義より上の3点を通る.

$y=a(x+1)(x-1)$

$\downarrow$ $(0,-\frac{1}{2})$通るので $a=\frac{1}{2}$

$y=\frac{1}{2}x^2-\frac{1}{2}$ //

(2)

$Q(X,Y)$, $P(t,\frac{t^2}{2}-\frac{1}{2})$とおくと

$$\begin{pmatrix}X\\Y\end{pmatrix}=\frac{1}{t^2+\frac{t^4}{4}-\frac{t^2}{2}+\frac{1}{4}}\begin{pmatrix}t\\\frac{t^2}{2}-\frac{1}{2}\end{pmatrix}$$

$$=\frac{1}{\left(\frac{t^2}{2}+\frac{1}{2}\right)^2}\begin{pmatrix}t\\\frac{t^2}{2}-\frac{1}{2}\end{pmatrix}$$

$\downarrow$

$$\begin{cases}X=\frac{4t}{(t^2+1)^2} & 奇関数\\ Y=\frac{2(t^2-1)}{(t^2+1)^2} & 偶関数\end{cases}$$

$$\frac{dX}{dt}=\frac{4(t^2+1)^2-4t\cdot2(t^2+1)2t}{(t^2+1)^4}=\frac{4(1-3t^2)}{(t^2+1)^3}$$

t	$\cdots$	$-\frac{1}{\sqrt{3}}$	$\cdots$	$\frac{1}{\sqrt{3}}$	$\cdots$
$\frac{dX}{dt}$	$-$	0	$+$	0	$-$
X	$\searrow$	$-\frac{3\sqrt{3}}{4}$	$\nearrow$	$\frac{3\sqrt{3}}{4}$	$\searrow$

$\lim_{t\to\pm\infty}X=0$ より小文字xに直して

$-\frac{3\sqrt{3}}{4}\leqq x\leqq\frac{3\sqrt{3}}{4}$ //

$$\frac{dY}{dt}=\frac{4t(t^2+1)^2-2(t^2-1)2(t^2+1)2t}{(t^2+1)^4}=\frac{-4t^3+12t}{(t^2+1)^3}=\frac{-4t(t+\sqrt{3})(t-\sqrt{3})}{(t^2+1)^3}$$

t	0	$\cdots$	$\sqrt{3}$	$\cdots$
$\frac{dY}{dt}$	0	$+$	0	$-$
Y	-2	$\nearrow$	$\frac{1}{4}$	$\searrow$

$\lim_{t\to\infty}Y=0$ より小文字yに直して

$-2\leqq y\leqq\frac{1}{4}$ //

(3) Cで$y\leqq0$のとき$-1\leqq t\leqq1$.

$$\int_{-1}^{1}\sqrt{\left(\frac{dX}{dt}\right)^2+\left(\frac{dY}{dt}\right)^2}\,dt$$

$$=\int_{-1}^{1}\sqrt{\frac{16(1-3t^2)^2}{(t^2+1)^6}+\frac{16t^2(t^2-3)^2}{(t^2+1)^6}}\,dt$$

$$=\int_{-1}^{1}\frac{1}{(t^2+1)^3}\sqrt{16(t^6+3t^4+3t^2+1)}\,dt$$

$$=\int_{-1}^{1}\frac{4}{(t^2+1)^3}\sqrt{(t^2+1)^3}\,dt$$

$$=8\int_0^1\frac{1}{\sqrt{(t^2+1)^3}}\,dt$$

$$=8\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{(\tan^2\theta+1)^3}}\frac{1}{\cos^2\theta}\,d\theta$$

$$=8\int_0^{\frac{\pi}{4}}\cos\theta\,d\theta$$

$=4\sqrt{2}$ //

(4)

$y=ax$にCを代入.

$$\frac{2(t^2-1)}{(t^2+1)^2}=a\cdot\frac{4t}{(t^2+1)^2}$$

$\Leftrightarrow t^2-2at-1=0$

この解をt_α, t_βとする.

(M_aのx座標)

$$=\frac{1}{2}\left\{\frac{4t_\alpha}{(t_\alpha^2+1)^2}+\frac{4t_\beta}{(t_\beta^2+1)^2}\right\}$$

$$=\frac{2t_\alpha}{(t_\alpha^2+1)^2}+\frac{2t_\beta}{(t_\beta^2+1)^2}$$

$$=\frac{2t_\alpha}{(2at_\alpha+1+1)^2}+\frac{2t_\beta}{(2at_\beta+1+1)^2}$$

$$=\frac{t_\alpha}{2(a^2t_\alpha^2+\underbrace{2at_\alpha+1}_{t_\alpha^2})}+\frac{t_\beta}{2(a^2t_\beta^2+\underbrace{2at_\beta+1}_{t_\beta^2})}$$

$$=\frac{t_\alpha}{2(a^2+1)t_\alpha^2}+\frac{t_\beta}{2(a^2+1)t_\beta^2}$$

$$=\frac{t_\alpha+t_\beta}{2(a^2+1)t_\alpha t_\beta}$$

$$=\frac{2a}{2(a^2+1)(-1)}=-\frac{a}{a^2+1}$$

$\therefore M_a\left(-\frac{a}{a^2+1},-\frac{a^2}{a^2+1}\right)$ //

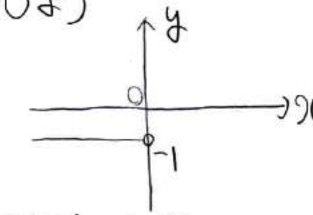
(5) $N_a(x,y)$とおくと

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{\frac{a^2}{(a^2+1)^2}+\frac{a^4}{(a^2+1)^2}}\overrightarrow{OM_a}$$

$$=\frac{(a^2+1)^2}{a^2(1+a^2)}\overrightarrow{OM_a}$$

$$=\frac{a^2+1}{a^2}\overrightarrow{OM_a}=\begin{pmatrix}-\frac{1}{a}\\-1\end{pmatrix}$$

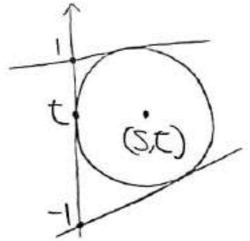
$a>0$より



求める軌跡は直線 $y=-1$ $(x<0)$ //

[IV]

(1)



$L_A : y = k_a x - 1$ とおく.

(s,t)との距離が$|s|$より

$$\frac{|k_a s - t - 1|}{\sqrt{k_a^2+1}} = |s|$$

分母はらって2乗すると

$(k_a s - t - 1)^2 = s^2(k_a^2+1)$

$\Leftrightarrow -2k_a s(t+1) + (t+1)^2 = s^2$

解くと $k_a = \dfrac{(t+1)^2 - s^2}{2s(1+t)}$ //

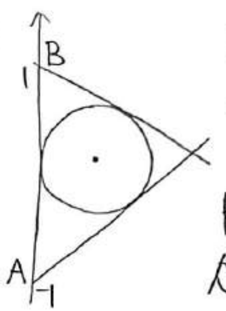
$L_B : y = k_b x + 1$ とおく.

同様に ここだけk_aと違う.

$$\frac{|k_b s - (t+1)|}{\sqrt{k_b^2+1}} = |s|$$

k_aの結果使うと $k_b = \dfrac{(t-1)^2 - s^2}{2s(-1+t)}$ //

(2)



$k_a x - 1 = k_b x + 1$

$\Leftrightarrow (k_a - k_b)x = 2$

$\Downarrow$

$k_a - k_b > 0$

が必要

$k_a - k_b > 0$

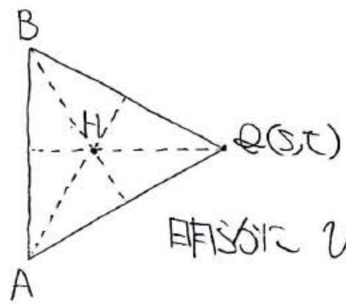
$\Leftrightarrow \dfrac{(t+1)^2 - s^2}{1+t} > \dfrac{(t-1)^2 - s^2}{t-1}$

(図より $-1 < t < 1$ に注意

$\Leftrightarrow \{(t+1)^2 - s^2\}(t-1) < \{(t-1)^2 - s^2\}(t+1)$

解くと $s^2 + t^2 < 1$ //

(3)



明らかに $v = t$

直線AQ : $y = \dfrac{t+1}{s}x - 1$

$\therefore$ BH : $y = -\dfrac{s}{t+1}x + 1$

これがH(u, v)通るので

$\underset{t}{v} = -\dfrac{s}{t+1}u + 1$

$\Leftrightarrow t^2 + t = -su + t + 1$

$\therefore u = \dfrac{1-t^2}{s}$ // $\quad v = t$ //

$L_A : k_a x - y - 1 = 0$ と R$(-u, -v)$

の距離は

$$d = \frac{|-k_a u + v - 1|}{\sqrt{k_a^2+1}}$$

(分子)

$= \left|-\dfrac{(t+1)^2 - s^2}{2s(1+t)} \cdot \dfrac{1-t^2}{s} + t - 1\right|$

$= \cdots = \dfrac{s^2 + (t+1)^2}{2s^2}(1-t)$

$(分母)^2$

$= k_a^2 + 1$

$= \dfrac{(1+t)^4 - 2s^2(1+t)^2 + (1+t)^4}{4s^2(1+t)^2} + 1$

$= \dfrac{\{(t+1)^2 + s^2\}^2}{4s^2(1+t)^2}$

$\therefore d = \dfrac{1-t^2}{s}$ //

(4)

Pは長軸が$2a$, 焦点がA, Bより

$$\frac{x^2}{a^2-1} + \frac{y^2}{a^2} = 1$$

上にある. Pのx座標をx_Pとすると

$x_P = \dfrac{2}{k_a - k_b} \quad y_P = k_a x_P - 1$

AP

$= \sqrt{x_P^2 + k_a^2 x_P^2}$

$= \sqrt{k_a^2+1}\,|x_P| = \dfrac{(t+1)^2 + s^2}{2s(1+t)}x_P$

同様に BP $= \dfrac{(t-1)^2 + s^2}{2s(t-1)}x_P$

AP+BP$=2a$より

$\left\{\dfrac{(1+t)^2 + s^2}{2s(1+t)} + \dfrac{s^2 + (1-t)^2}{2s(1-t)}\right\}x_P = 2a$

$\Leftrightarrow \cdots \Leftrightarrow \dfrac{1-t^2+s^2}{s(1-t^2)}x_P = 2a \cdots ①$

$k_a - k_b$

$= \dfrac{(t+1)^2 - s^2}{t+1} - \dfrac{(t-1)^2 - s^2}{t-1}$

$= \cdots = \dfrac{1-s^2-t^2}{s(1-t^2)}$

$\therefore x_P = \dfrac{2}{k_a - k_b} = \dfrac{2s(1-t^2)}{1-s^2-t^2}$

①に代入して

$\dfrac{2(1-t^2+s^2)}{1-s^2-t^2} = 2a$

$\Leftrightarrow 1-t^2+s^2 = a(1-s^2-t^2)$

$\Leftrightarrow \cdots$

$\Leftrightarrow \dfrac{a+1}{a-1}s^2 + t^2 = 1$

Qが動く2次曲線は

$\underline{\dfrac{a+1}{a-1}}x^2 + \underline{0}\cdot x + \underline{1}\cdot y^2 + \underline{0}\cdot y = 1$ // // // //