2019 東京慈恵会医大

1.

(1)

P(3回目で初めて硬貨もらう)

$= P(3回目で初めて x+y=3)$

$= \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \underline{\frac{9}{64}}$ //

P(4回目で2枚目の硬貨もらう)

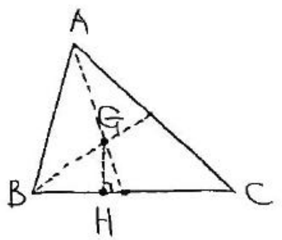
$= P(2回目と4回目で x+y=0)$

$+P(1回目で x+y=3,\ 4回目で x+y=0)$

$= \left(\frac{1}{4}\right)^2 {}_2C_1 \frac{3}{4}\cdot\frac{1}{4} + \frac{3}{4}\left(\frac{1}{4}\right)^3$

$= \underline{\frac{9}{256}}$ //

(2)



余弦定理より

$\cos B = \frac{16+25-36}{2\cdot 4\cdot 5}$

$= \frac{1}{8}$

これより

B(0,0), C(5,0)

$A(\underbrace{4\cos B}_{\frac{1}{2}}, 4\sin B)$ とおける.

Gの x 座標は $\frac{\frac{1}{2}+5}{3} = \frac{11}{6}$

$\therefore \frac{BH}{BC} = \frac{\frac{11}{6}}{5} = \underline{\frac{11}{30}}$ //

2.

(1) $y = \frac{3e^x-1}{e^x+1}$

$y' = \frac{3e^x(e^x+1)-(3e^x-1)e^x}{(e^x+1)^2}$

$= \frac{4e^x}{(e^x+1)^2}$

$y'' = \frac{4e^x(e^x+1)^2-8e^x(e^x+1)e^x}{(e^x+1)^4}$

$= \frac{4e^x\{e^x+1-2e^x\}}{(e^x+1)^3}$

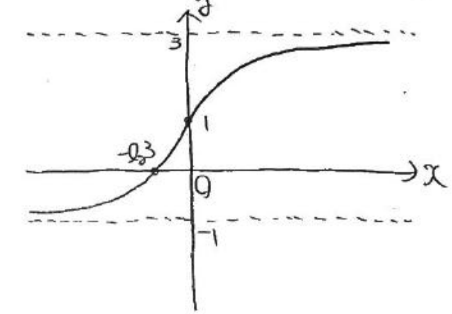
$= \frac{4e^x(1-e^x)}{(e^x+1)^3}$

$x>0$ で $y''<0$ つまり上に凸

$x<0$ で $y''>0$ つまり下に凸

変曲点 (0, 1)

$\lim_{x\to\infty} y = 3$, $\lim_{x\to-\infty} y = -1$



(2) Pの x 座標を s とおく.

$\begin{cases} 傾き一致: \frac{4e^s}{(e^s+1)^2} = \frac{e^s}{a^2} \\ y座標一致: \frac{3e^s-1}{e^s+1} = \frac{e^s}{a^2}+b \end{cases}$

$a^2 = \frac{(e^s+1)^2}{4}$

$\therefore a = \frac{e^s+1}{2} \iff e^s = 2a-1$

Pの座標は

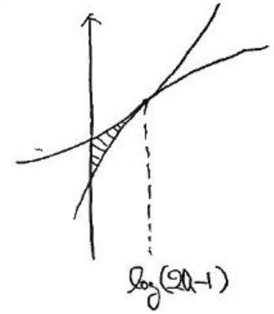
$\left(\log(2a-1), \frac{6a-4}{2a}\right)$

$\therefore \underline{\left(\log(2a-1), 3-\frac{2}{a}\right)}$ //

$\frac{2a-1}{a^2}+b = 3-\frac{2}{a}$

$\therefore \underline{b = 3-\frac{4}{a}+\frac{1}{a^2}}$ //

(3)



$S(a)$

$= \int_0^{\log(2a-1)}\left(\frac{e^x}{a^2}+b-\frac{3e^x-1}{e^x+1}\right)dx$

$= \int_0^{\log(2a-1)}\left(\frac{e^x}{a^2}+b-\frac{-e^x-1+4e^x}{e^x+1}\right)dx$

$= \int_0^{\log(2a-1)}\left(\frac{e^x}{a^2}+b+1-\frac{4e^x}{e^x+1}\right)dx$

$= \left[\frac{e^x}{a^2}+(b+1)x-4\log(e^x+1)\right]_0^{\log(2a-1)}$

$= \frac{2a-1}{a^2}+(b+1)\log(2a-1)-4\log 2a$

$-\left(\frac{1}{a^2}-4\log 2\right)$

$= \frac{2}{a}-\frac{1}{a^2}+\left(4-\frac{4}{a}+\frac{1}{a^2}\right)\log(2a-1)$

$-4\log a-\frac{1}{a^2}$

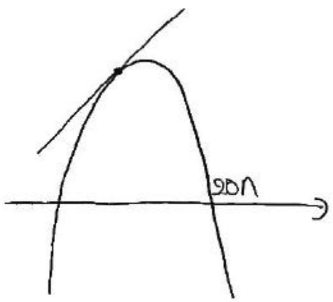
$= \frac{2}{a}-\frac{2}{a^2}+\left(4-\frac{4}{a}+\frac{1}{a^2}\right)\log\left(2-\frac{1}{a}\right)$

$+\left(4-\frac{4}{a}+\frac{1}{a^2}\right)\log a-4\log a$

$$= \frac{2}{a} - \frac{2}{a^2} + \left(4 - \frac{4}{a} + \frac{1}{a^2}\right)\log\left(2 - \frac{1}{a}\right) + \left(-4 + \frac{1}{a}\right)\frac{\log a}{a}$$

$$\therefore \lim_{a \to \infty} S(a) = \underline{4\log 2}$$ //

3. $y - x = k$ とおく



図のように接するのは

$-x^2 + 20nx = x + k$

$\Leftrightarrow 0 = x^2 + (1-20n)x + k$

$D = (1-20n)^2 - 4k = 0$

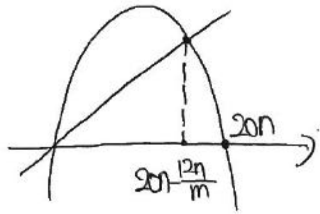
このとき 交点 $x = \dfrac{20n-1}{2} = 10n - \dfrac{1}{2}$

ところで $y = -x^2 + 20nx$ の原点での接線の傾きが $20n$ であり

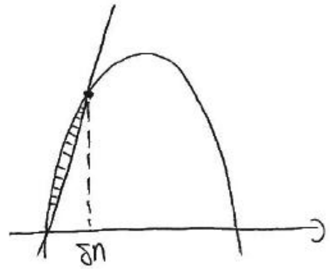
$\dfrac{12n}{m} - 20n = 4n\left(\dfrac{3}{m} - 5\right) < 0$

より $\dfrac{12n}{m} < 20n$ から.

$y = -x^2 + 20nx$ と $y = \dfrac{12n}{m}x$ は原点以外で第1象限で交わる.



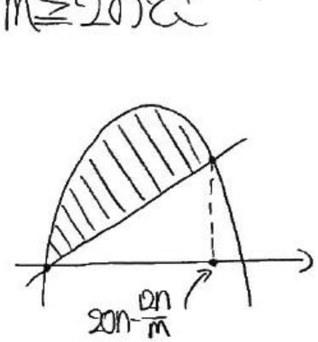
(i) $m = 1$ のとき



このとき $x = 8n$, $y = 96n^2$ で

$\max k = 96n^2 - 8n$ //

(ii) $m \geqq 2$ のとき



$x = 10n - \dfrac{1}{2}$ は自然数でない.

$x = 10n$, $y = 100n^2$ のとき

$k = 100n^2 - 10n$

$x = 10n - 1$, $y = (10n-1)(10n+1) = 100n^2 - 1$ のとき

$k = 100n^2 - 1 - (10n - 1)$

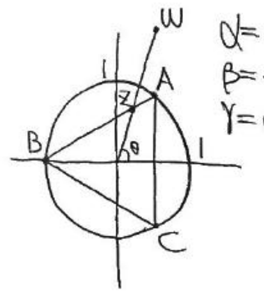
$= 100n^2 - 10n$

どちらも同じなのでこれらの (x, y) で

$\max k = 100n^2 - 10n$ //

4.

(1)



$\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\beta = -1$

$\gamma = \cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi$

$$\begin{cases} z = |z|(\cos\theta + i\sin\theta) \\ w = |w|(\cos\theta + i\sin\theta) \end{cases}$$ とおく

$\bar{z} = |z|\{\cos(-\theta) + i\sin(-\theta)\}$

$= |z|(\cos\theta - i\sin\theta)$

$w\bar{z} = |w||z|(\cos^2\theta + \sin^2\theta)$

$= 1 \quad (\because |w| = \frac{1}{|z|})$

$\therefore w = \dfrac{1}{\bar{z}}$

(2)

z が線分AC上にあるとき

$z = \dfrac{1}{2} + ki \quad \left(-\dfrac{\sqrt{3}}{2} \leqq k \leqq \dfrac{\sqrt{3}}{2}\right)$

とおける.

$w = \dfrac{1}{\bar{z}}$

$= \dfrac{1}{\frac{1}{2} - ki}$

$= \dfrac{2}{1-2ki} \times \dfrac{1+2ki}{1+2ki}$

$= \dfrac{2}{1+4k^2} + \dfrac{4k}{1+4k^2}i$

$= X + Yi$ とすると

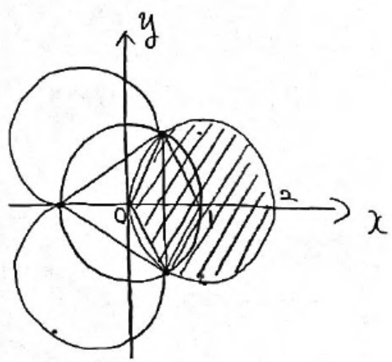
$$\begin{cases} X = \dfrac{2}{1+4k^2} \\ Y = \dfrac{4k}{1+4k^2} \end{cases}$$

$X^2 + Y^2 = \dfrac{4 + 16k^2}{(1+4k^2)^2} = \dfrac{4}{1+4k^2} = 2X$

$\Leftrightarrow (X-1)^2 + Y^2 = 1$

$-\dfrac{\sqrt{3}}{2} \leqq k \leqq \dfrac{\sqrt{3}}{2}$ より $\dfrac{1}{2} \leqq X \leqq 2$

対称性を考えて図を書く



図の斜線部分を3倍すればいい.

(斜線部分)

$=$  $+$ 

$= 1 \times \frac{\sqrt{3}}{2} + 1 \cdot 1 \cdot \pi \cdot \frac{2}{3}$

$= \frac{2}{3}\pi + \frac{\sqrt{3}}{2}$

$\therefore \underline{S = 2\pi + \frac{3\sqrt{3}}{2}}$ //