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[I]

(1) $q^3 = 35 - p^3 \geqq 8$

$\therefore 8 \leqq p^3 \leqq 27$

$\therefore p = 2, 3$

$\therefore (p, q) = (2, 3), (3, 2)$

(2) $\sqrt{\dfrac{1+x}{2}} = 1 - 2x^2 \geqq 0$

$\therefore -\dfrac{1}{\sqrt{2}} \leqq x \leqq \underbrace{\dfrac{1}{\sqrt{2}}}_{\fallingdotseq 0.7} \cdots ①$

2乗すると

$\dfrac{1+x}{2} = 4x^4 - 4x^2 + 1$

$\Leftrightarrow 0 = 8x^4 - 8x^2 - x + 1$

8	0	−8	−1	1	
	8	8	0	−1	1
8	8	0	−1		

$\Leftrightarrow 0 = (x-1)(8x^3 + 8x^2 - 1)$

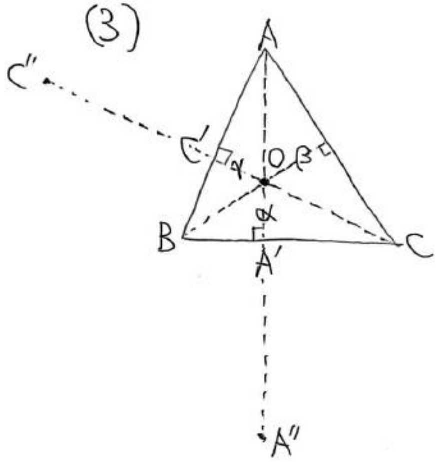
$\Leftrightarrow 0 = (x-1)(2x+1)(4x^2 + 2x - 1)$

$\Leftrightarrow x = 1, -\dfrac{1}{2}, \dfrac{-1 \pm \sqrt{5}}{4}$

①より求める答えは

$x = -\dfrac{1}{2}, \dfrac{-1+\sqrt{5}}{4}$ //

(3)



$\triangle OAB : \triangle OBC = OA : OC = \gamma : \alpha$

$\downarrow$

$OA = \gamma t,\ OC = \alpha t$ とおく.

四角形$ABOC = \triangle OBC = OA : OA' = \gamma + \beta : \alpha$

$OA' = \dfrac{\alpha}{\gamma+\beta} OA = \dfrac{\alpha\gamma}{\gamma+\beta} t$

同様に

$OC' = \dfrac{\gamma}{\beta+\alpha} OC = \dfrac{\alpha\gamma}{\alpha+\beta} t$

以上より

$\triangle OA''C'' = \dfrac{OA''}{OA} \cdot \dfrac{OC''}{OC} \triangle OAC$

$= \dfrac{2OA' + OA}{\gamma t} \cdot \dfrac{2OC' + OC}{\alpha t} \beta$

$= \left(\dfrac{2\alpha}{\gamma+\beta} + 1\right)\left(\dfrac{2\gamma}{\alpha+\beta} + 1\right)\beta$

$= \dfrac{(2\alpha+\beta+\gamma)(\alpha+\beta+2\gamma)\beta}{(\beta+\gamma)(\alpha+\beta)}$

(4) $\log_{10} 6^{74}$

$= 74 \log_{10} 6$

$= 57.5868$

$\downarrow$

$57 < \log_{10} 6^{74} < 58$

$\Leftrightarrow 10^{57} < 6^{74} < 10^{58}$

6^{74}は58桁 //

また

$\log_{10} 3 < 0.5868 < \log_{10} 4$

$\Leftrightarrow 57 + \log_{10} 3 < 57.5868 < 57 + \log_{10} 4$

$\Leftrightarrow \log_{10} 3 \cdot 10^{57} < \log_{10} 6^{74} < \log_{10} 4 \cdot 10^{57}$

$\Leftrightarrow 3 \cdot 10^{57} < 6^{74} < 4 \cdot 10^{57}$

6^{74}の最高位の数字は3 //

[II] (1)

$f(x) = x^4 - x^3 + x^2 - x + 1$

$f'(x) = 4x^3 - 3x^2 + 2x - 1$

$f''(x) = 12x^2 - 6x + 2$

$f''(x) = 0$の判別式をDとおくと

$\dfrac{D}{4} = 9 - 24 < 0$

$f''(x) = 12\left(x - \dfrac{1}{4}\right)^2 + \dfrac{5}{4} > 0$

これより $f'(x)$は単調増加.

$f'(0) = -1,\quad f'(1) = 2$

より $0 < x < 1$の間に実数解を1つもつ.

(2)

$f'\left(\dfrac{1}{2}\right) = -\dfrac{1}{4} < 0$

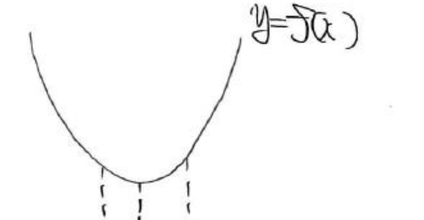
$f'\left(\dfrac{3}{4}\right) = \dfrac{27}{16} - \dfrac{27}{16} + \dfrac{3}{2} - 1$

$= \dfrac{1}{2} > 0$

求める答えは

$a = \dfrac{1}{2}$ //

(3)



$y=f(x)$は$x=x_0$で極小値かつ最小値．$f\left(\frac{1}{2}\right)=\frac{11}{16}$より

$$f(x_0)<\frac{11}{16}.$$

また

$f'(x_0)=4x_0^3-3x_0^2+2x_0-1=0$

より

$$\begin{array}{r|rrrrr}
 & \frac{1}{4} & -\frac{1}{16} & & & \\
\hline
4\ -3\ \ 2\ -1 & 1 & -1 & 1 & -1 & 1 \\
 & 1 & -\frac{3}{4} & \frac{1}{2} & -\frac{1}{4} & \\
\hline
 & & -\frac{1}{4} & \frac{1}{2} & -\frac{3}{4} & 1 \\
 & & -\frac{1}{4} & \frac{3}{16} & -\frac{1}{8} & \frac{1}{16} \\
\hline
 & & & \frac{5}{16} & -\frac{5}{8} & \frac{15}{16}
\end{array}$$

$f(x_0)$

$=(4x_0^3-3x_0^2+2x_0-1)\left(\frac{1}{4}x_0-\frac{1}{16}\right)$

$\qquad\frac{5}{16}x_0^2-\frac{5}{8}x_0+\frac{15}{16}$

$=\frac{5}{16}(x_0^2-2x_0+3)$

$=\frac{5}{16}\{(x_0-1)^2+2\}>\frac{5}{8}$ $(x_0\neq 1)$

$\therefore\ \underline{\frac{5}{8}<f(x_0)<\frac{11}{16}}$ //

[III]

(1) $n=pq$ (p, qは2以上の自然数) とする．

$n^2=p^2q^2$の約数dを

$$d=p^2q$$

とおくと $p^2q>pq=n$ より

$n<d<n^2$をみたすdが存在．

(2)

$\sum_{k=0}^{50}{}_{101}C_k$

$=\frac{1}{2}\sum_{k=0}^{101}{}_{101}C_k$

$=\frac{1}{2}(1+1)^{101}$

$=2^{100}$

これの約数の個数は負の数も含めると$\underline{202個}$ //

(3)

$X=3^4\cdot\frac{1}{2}\sum_{k=1}^{11}{}_{11}C_k$

$=3^4\cdot\frac{1}{2}\cdot 2^{11}=2^{10}\cdot 3^4$

(ア) $D=\overline{A}\cap B\cap C$

(イ) $X^2=2^{20}\cdot 3^8$

$d\in D$であるdは

$d=\{2^{11}\cdot 3^3, 2^{11}\cdot 3^2, 2^{12}\cdot 3^2,$

$2^{13}\cdot 3^2, 2^{11}\cdot 3, 2^{12}\cdot 3,$

$2^{13}\cdot 3, 2^{14}\cdot 3,$

$2^k\ (k=11\sim 16)$

$2^a\cdot 3^5\ (a=0\sim 8)$

$2^b\cdot 3^6\ (b=0\sim 6)$

$2^c\cdot 3^7\ (c=0\sim 5)$

$2^d\cdot 3^8\ (d=0\sim 3)\}$

これらすべてで$\underline{40個}$ //

[IV]

(1) $\frac{1}{\cos^3 x}$

(2)

$\int\frac{1}{\cos x}dx$

$=\int\frac{\cos x}{1-\sin^2 x}dx$

$=\int\left(\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}\right)\frac{1}{2}dx$

$=\frac{1}{2}\{-\log|1-\sin x|+\log|1+\sin x|\}+C$

$=\frac{1}{2}\log\frac{1+\sin x}{1-\sin x}+C.$

(Cは積分定数)

(3)

$\int\frac{1}{\cos^{2n+1}x}dx$

$=\int\frac{1}{\cos^2 x}\cdot\frac{1}{\cos^{2n-1}x}dx$

$=\tan x\cdot\frac{1}{\cos^{2n-1}x}$

$\qquad-\int\tan x\frac{-\sin x}{\cos^{2n}x}(1-2n)dx$

$=\frac{\tan x}{\cos^{2n-1}x}$

$\qquad-\int\tan^2 x\frac{2n-1}{\cos^{2n-1}x}dx$

$=\frac{\tan x}{\cos^{2n-1}x}$

$\qquad-(2n-1)\int\left(\frac{1}{\cos^2 x}-1\right)\frac{1}{\cos^{2n-1}x}dx$

$\Leftrightarrow$

$2n\int\frac{1}{\cos^{2n+1}x}dx$

$=\frac{\tan x}{\cos^{2n-1}x}+(2n-1)\int\frac{1}{\cos^{2n-1}x}dx$

(4)

(3)で$n=1$を代入すると

$$2\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3 x}dx$$

$$= \left[\frac{\tan x}{\cos x}\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{\cos x}dx$$

$$= \sqrt{2} + \left[\frac{1}{2}\log\frac{1+\sin x}{1-\sin x}\right]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2} + \frac{1}{2}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \sqrt{2} + \log(\sqrt{2}+1)$$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{1}{\cos^3 x}dx = \underline{\frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2}+1)}$$

[V]

(1)

(ア)

$$P(X=10n) = p^n$$

$$P(X=10n-5) = p^{n-1}(1-p) \times {}_nC_1$$

$$= np^{n-1}(1-p)$$

⋮

$$\downarrow n=3,\ p=0.5$$

$$P(X=30) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$P(X=25) = 3\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$P(X=20) = 3\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$\underline{P(X=15) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}}$$

(イ)

$$E(X)$$

$$= \frac{30+75+60+15}{8}$$

$$= \frac{180}{8} = \underline{\frac{45}{2}}$$

$$V(X)$$

$$= E(X^2) - (E(X))^2$$

$$= \frac{900+1875+1200+225}{8} - \frac{2025}{4}$$

$$= \frac{4200}{8} - \frac{4050}{8}$$

$$= \frac{150}{8} = \underline{\frac{75}{4}}$$

(2) (ア)

$$P(M=k) = {}_nC_k\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k}$$

$$= {}_nC_k\left(\frac{1}{2}\right)^n$$

(イ) $X = 10M + 5(n-M)$

$$= 5M + 5n$$

(ウ)

$$E(X)$$

$$= E(5M+5n)$$

$$= 5\cdot E(M) + 5n$$

$$= 5\cdot\sum_{k=0}^{n} k\cdot{}_nC_k\left(\frac{1}{2}\right)^n + 5n$$

$$= \frac{5}{2^n}\cdot\sum_{k=0}^{n} k\cdot{}_nC_k + 5n$$

ここで

$$(1+x)^n = \sum_{k=0}^{n} {}_nC_k x^k$$

両辺xで微分すると

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\cdot{}_nC_k x^{k-1} \cdots ①$$

$$\downarrow x=1$$

$$n\cdot 2^{n-1} = \sum_{k=0}^{n} k\cdot{}_nC_k$$

$$= \frac{5}{2}n + 5n = \underline{\frac{15}{2}n}$$

$$V(X)$$

$$= E(X^2) - (E(X))^2$$

$$= E(25M^2 + 50nM + 25n^2) - \frac{225}{4}n^2$$

$$= 25E(M^2) + 50n\cdot E(M) + 25n^2 - \frac{225}{4}n^2$$

ここで①を両辺xで微分すると

$$n(n-1)(1+x)^{n-2} = \sum_{k=0}^{n} k(k-1){}_nC_k x^{k-2}$$

$$\downarrow x=1$$

$$n(n-1)2^{n-2} = \sum_{k=0}^{n} k(k-1){}_nC_k$$

$$\therefore \sum_{k=1}^{n} k^2\cdot{}_nC_k = n(n-1)2^{n-2} + n\cdot 2^{n-1}$$

$$= n(n+1)2^{n-2}$$

$$V(X)$$

$$= 25n(n+1)\frac{1}{4} + 50n\cdot\frac{n}{2} - \frac{125}{4}n^2$$

$$= \frac{25}{4}n(n+1) - \frac{25}{4}n^2 = \frac{25}{4}n$$

$$\therefore \sigma(X) = \underline{\frac{5}{2}\sqrt{n}}$$

(エ)

$$\frac{\sigma(X)}{E(X)}$$

$$= \frac{\frac{5}{2}\sqrt{n}}{\frac{15}{2}n}$$

$$= \frac{1}{3\sqrt{n}} < \frac{1}{10}$$

$$\Leftrightarrow \sqrt{n} > \frac{10}{3}$$

$$\therefore n > \frac{100}{9}$$

求めるnの最小値は12.

(3)

(ア)

$P(M \geqq 64)$

$= P(X \geqq 5\cdot 64 + 500)$

$= P(\underbrace{X - 750}_{E(X)} \geqq 70)$

$= P\left(\dfrac{X-750}{\underbrace{25}_{\sigma(X)}} \geqq \underbrace{\dfrac{14}{5}}_{2.8}\right)$

$= 0.5 - 0.49744$

$= 0.00256$

$\fallingdotseq \underline{0.002}$ //

(イ)

nが大きいとき$R = \dfrac{M}{n}$は近似的に$N\left(p, \dfrac{p(1-p)}{n}\right)$に従う.

$$P\left(\left|\frac{R-p}{\sqrt{\frac{p(1-p)}{n}}}\right| \leqq 1.96\right) = 0.95$$

$\Downarrow$

$$-1.96\sqrt{\frac{p(1-p)}{n}} \leqq R - p \leqq 1.96\sqrt{\frac{p(1-p)}{n}}$$

$$\Leftrightarrow R - 1.96\sqrt{\frac{p(1-p)}{n}} \leqq p \leqq R + 1.96\sqrt{\frac{p(1-p)}{n}}$$

nが十分大きいとき95%信頼区間は

$$R - 1.96\sqrt{\frac{R(1-R)}{n}} \leqq p \leqq R + 1.96\sqrt{\frac{R(1-R)}{n}}$$

$\Downarrow$

$$0.64 - \underbrace{1.96\sqrt{\frac{0.64\times 0.36}{100}}}_{0.09408} \leqq p \leqq 0.64 + 1.96\sqrt{\frac{0.64\times 0.36}{100}}$$

$\Downarrow$ 小数点以下第3位

$\underline{0.545 \leqq p \leqq 0.734}$ //

(ウ)

(95%信頼区間の幅)

$= 2\times 1.96 \times \sqrt{\dfrac{R(1-R)}{n}} \leqq 0.1$

$\Leftrightarrow 39.2 \times \sqrt{R(1-R)} \leqq \sqrt{n}$

$\Downarrow$ 2乗

$n \geqq 1536.64R(1-R)$

$= 1536.64\left\{-\left(R-\frac{1}{2}\right)^2 + \frac{1}{4}\right\}$

どんなRでも上の不等式が成り立つ最小のnは $1536.64 \times \frac{1}{4} = 384.16$ より

$\underline{385}$ //