2018 上智大理工

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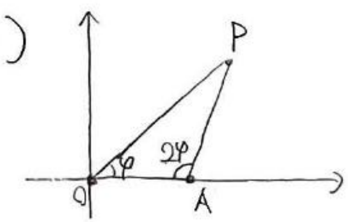
(1)



$P(x, y)$ とおくと

$$\begin{cases} \tan\varphi = \dfrac{y}{x} \quad \cdots ① \\ \tan(\pi - 2\varphi) = \dfrac{y}{x-1} \end{cases}$$

$\hookrightarrow \tan 2\varphi = \dfrac{y}{1-x}$

$\Longleftrightarrow \dfrac{2\tan\varphi}{1-\tan^2\varphi} = \dfrac{y}{1-x}$

$\downarrow ①$

$$\frac{\frac{2y}{x}}{1-\frac{y^2}{x^2}} = \frac{y}{1-x}$$

$\Longleftrightarrow \dfrac{2x}{x^2-y^2} = \dfrac{1}{1-x} \quad (y>0)$

$\Longleftrightarrow x^2 - y^2 = -2x^2 + 2x$

$\Longleftrightarrow 3x^2 - 2x - y^2 = 0$

$\Longleftrightarrow 3\left(x-\frac{1}{3}\right)^2 - y^2 = \frac{1}{3}$

$\Longleftrightarrow 9\left(x-\frac{1}{3}\right)^2 - 3y^2 = 1$

$\therefore \underline{9}\left(x+\underline{\frac{-1}{3}}\right)^2 + \underline{(-3)}y^2 = 1$

$\therefore \left(\dfrac{x-\frac{1}{3}}{\frac{1}{3}}\right)^2 - \left(\dfrac{y}{\frac{1}{\sqrt{3}}}\right)^2 = 1$

漸近線 $y = \dfrac{\frac{1}{\sqrt{3}}}{\frac{1}{3}}\left(x+\frac{1}{3}\right)$

$= \underline{\sqrt{3}}\left(x+\underline{\frac{1}{3}}\right)$

(2)

$PH = |k-x|$

$AP = \sqrt{(x-1)^2+y^2}$

$\downarrow$

$\dfrac{PH}{AP} = \dfrac{|k-x|}{\sqrt{(x-1)^2+y^2}} = \alpha$ とおくと

$\downarrow$ 2乗して整理

$x^2 - 2kx + k^2 = \alpha^2(x-1)^2 + \alpha^2 y^2$

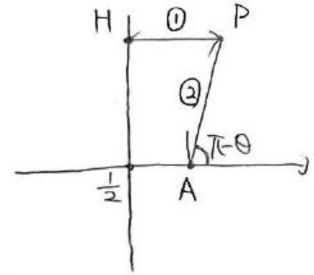
$\Longleftrightarrow (1-\alpha^2)x^2 + (2\alpha^2 - 2k)x - \alpha^2 y^2 + \underbrace{k^2 - \alpha^2}_{0} = 0$

$\downarrow$

$\dfrac{1-\alpha^2}{\alpha^2} = 3 \qquad \therefore \alpha = \dfrac{1}{2}$

$k = \dfrac{1}{2}$

$\underline{k = \frac{1}{2}}, \quad \underline{\dfrac{PH}{AP} = \dfrac{1}{2}}$



図より

$\dfrac{1}{2} + PA\cos(\pi-\theta) = PH$

$\downarrow PH = \frac{1}{2}PA$

$\dfrac{1}{2} - PA\cos\theta = \dfrac{1}{2}PA$

$\Longleftrightarrow 1 - 2PA\cos\theta = PA$

$\therefore PA = \underline{\dfrac{1}{1+2\cos\theta}}$

2

(1) n日目にA定食が提供される確率を P_n とおくと

	n日目		$n+1$日目
A定食	P_n	$\xrightarrow{\frac{1}{2}}$	P_{n+1}
B定食	$1-P_n$	$\nearrow 1$	

$P_{n+1} = \dfrac{1}{2}P_n + 1 - P_n$

$= -\dfrac{1}{2}P_n + 1$

$\Longleftrightarrow P_{n+1} - \dfrac{2}{3} = -\dfrac{1}{2}\left(P_n - \dfrac{2}{3}\right)$

$\therefore P_n - \dfrac{2}{3} = \left(\underbrace{P_1}_{1} - \dfrac{2}{3}\right)\left(-\dfrac{1}{2}\right)^{n-1}$

$\therefore P_n = \dfrac{1}{3}\left(-\dfrac{1}{2}\right)^{n-1} + \dfrac{2}{3}$

$P_3 = \dfrac{1}{3}\cdot\dfrac{1}{4} + \dfrac{2}{3} = \dfrac{9}{12} = \underline{\dfrac{3}{4}}$

(2)

$P_n = \underline{\dfrac{1}{3}}\left(\underline{\left(\dfrac{-1}{2}\right)}^{n-1} + \underline{2}\right)$

(3) 3　4　5　6 (日目)

$$A \begin{cases} A \begin{cases} A \begin{cases} A \\ B \end{cases} \\ B - A \end{cases} \\ B - A \begin{cases} A \\ B \end{cases} \end{cases}$$

P(3日目と6日目がA定食)

$= P_3 \times \left\{\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 \cdot 1 \cdot 2\right\}$

$= \dfrac{3}{4} \times \dfrac{5}{8}$

求める確率は

$P_{6日目がA}$(3日目がA)

$= \dfrac{\frac{3}{4}\times\frac{5}{8}}{P_6}$

$= \dfrac{\frac{3}{4}\cdot\frac{5}{8}}{\frac{1}{3}\left(-\frac{1}{32}+2\right)} = \underline{\dfrac{5}{7}}$

(4) n日目がB定食であるものの総数を b_n とおくと

$$\begin{cases} a_{n+1} = a_n + b_n \\ b_{n+1} = a_n \end{cases}$$

↓

$$a_{n+2} = a_{n+1} + a_n$$

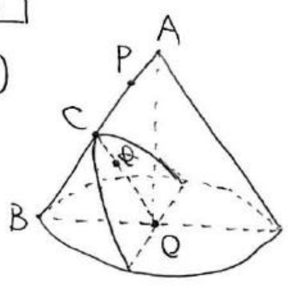
↓

$a_1 = 1$	$a_6 = 8$
$a_2 = 1$	$a_7 = 13$
$a_3 = 2$	$a_8 = 21$
$a_4 = 3$	$a_9 = 34$
$a_5 = \underline{5}$	$a_{10} = 55$

$\therefore a_{10} + b_{10} = a_{10} + a_9 = \underline{89}$

3

(1)



円錐面の方程式

$$x^2 + y^2 = (\sqrt{2} - z)^2$$

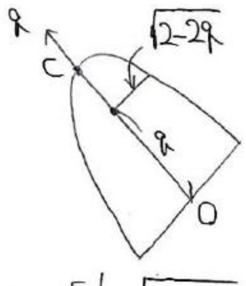
線分OCに $OQ = q$ $(0 \leqq q \leqq \sqrt{2})$ となるようにとると

$$Q\left(-\frac{q}{\sqrt{2}}, 0, \frac{q}{\sqrt{2}}\right).$$

$x = -\frac{q}{\sqrt{2}}$, $z = \frac{q}{\sqrt{2}}$ を円錐面の方程式に代入すると

$$\frac{q^2}{2} + y^2 = 2 - 2\sqrt{2}\frac{q}{\sqrt{2}} + \frac{q^2}{2}$$

$$\Longleftrightarrow y^2 = 2 - 2q$$



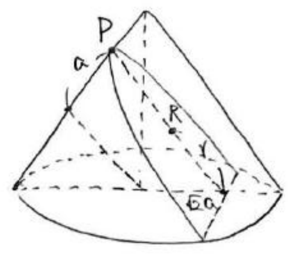
$$S(0) = \int_0^1 2\sqrt{2-2q}\,dq$$

$$= \left[2 \cdot \frac{2}{3}(2-2q)^{\frac{3}{2}}\left(-\frac{1}{2}\right)\right]_0^1$$

$$= \frac{4}{3} 2\sqrt{2} \cdot \frac{1}{2}$$

$$= \underline{\frac{4}{3}\sqrt{2}}$$

(2)



図のようにRをとると

$$R\left(\sqrt{2}a - \frac{r}{\sqrt{2}}, 0, \frac{r}{\sqrt{2}}\right)$$

$x = \sqrt{2}a - \frac{r}{\sqrt{2}}$, $z = \frac{r}{\sqrt{2}}$ を円錐面の方程式に代入すると

$$2a^2 - 2ar + \frac{r^2}{2} + y^2 = 2 - 2r + \frac{r^2}{2}$$

$$\Longleftrightarrow y^2 = 2 - 2a^2 - 2(1-a)r$$

同様に

$$S(a) = \int_0^{a+1} 2\sqrt{2 - 2a^2 - 2(1-a)r}\,dr$$

$$= 2\sqrt{2-2a}\int_0^{a+1}\sqrt{1+a-r}\,dr$$

$$= 2\sqrt{2-2a}\left[\frac{2}{3}(1+a-r)^{\frac{3}{2}}(-1)\right]_0^{a+1}$$

$$= \frac{4}{3}\sqrt{2-2a}\,(1+a)^{\frac{3}{2}}$$

$$= \frac{4\sqrt{2}}{3}(1-a)^{\frac{1}{2}}(1+a)^{\frac{3}{2}}$$

$S'(a)$

$$= \frac{4\sqrt{2}}{3}\left\{-\frac{1}{2}(1-a)^{-\frac{1}{2}}(1+a)^{\frac{3}{2}} + \frac{3}{2}(1-a)^{\frac{1}{2}}(1+a)^{\frac{1}{2}}\right\}$$

$$= \frac{4\sqrt{2}}{3}(1-a)^{-\frac{1}{2}}(1+a)(1-2a)$$

a	$0 \cdots \frac{1}{2} \cdots 1$
$S'(a)$	$+ \quad 0 \quad -$
$S(a)$	$\nearrow \quad \searrow$

$$\max S(a) = S\left(\frac{1}{2}\right) = \frac{4\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} \cdot \frac{3\sqrt{3}}{2\sqrt{2}} = \underline{\sqrt{6}}$$

(3) 求める体積は

$$\int_0^1 S(a)\,da$$

$$= \int_0^1 \frac{4\sqrt{2}}{3}(1-a)^{\frac{1}{2}}(1+a)^{\frac{3}{2}}\,da$$

$$= \int_0^1 \frac{4\sqrt{2}}{3}\sqrt{1-a^2}\,(1+a)\,da$$

↓ $a = \sin\theta$ とおく

$$= \int_0^{\frac{\pi}{2}} \frac{4\sqrt{2}}{3}\cos\theta(1+\sin\theta)\cos\theta\,d\theta$$

$$= \frac{4\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}\left\{\frac{1+\cos 2\theta}{2} + \cos^2\theta\sin\theta\right\}d\theta$$

$$= \frac{4\sqrt{2}}{3}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta - \frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{4\sqrt{2}}{3}\left(\frac{\pi}{4} + \frac{1}{3}\right)$$

$$= \underline{\frac{\sqrt{2}}{3}\pi + \frac{4}{9}\sqrt{2}}$$

$\boxed{4}$

(1)

$PA^2 \cdot PB^2$

$= \{(x-a)^2+(y-a)^2\}\{(x+a)^2+(y+a)^2\}$

$= \{x^2+y^2+2a^2-2a(x+y)\}$
$\quad \{x^2+y^2+2a^2+2a(x+y)\}$

$= (x^2+y^2+2a^2)^2-4a^2(x+y)^2$

$= (x^2+y^2)^2+4a^2(x^2+y^2)+4a^4$
$\quad -4a^2(x^2+y^2+2xy)$

$= 2xy+4a^4-4a^2 \cdot 2xy$

$= 2xy\underbrace{(1-4a^2)}_{0}+4a^4$

$\therefore a=\frac{1}{2} \quad (a>0)$

$PA \cdot PB = \sqrt{4a^4} = \frac{1}{2}$ //

(2) $x=r\cos\theta,\ y=r\sin\theta$ を代入.

$r^4 = 2r^2\cos\theta\sin\theta$

$\Leftrightarrow r^2 = \sin 2\theta$

$\therefore s=2,\ t=2$ //

(3) 第2, 4象限だと $\sin 2\theta < 0$ になり定義できない. $\therefore$ F //

(4) x^2で考える. $0 \leqq \theta \leqq \frac{\pi}{2}$ を考える.

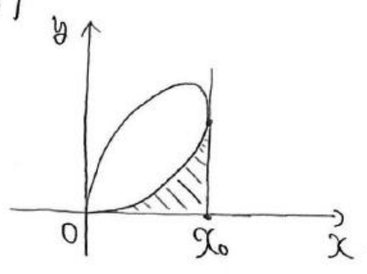
x^2
$= r^2\cos^2\theta$
$= \sin 2\theta \cdot \cos^2\theta$
$= 2\sin\theta\cos^3\theta$

$\frac{dx^2}{d\theta} = 2\cos^4\theta - 6\sin^2\theta\cos^2\theta$
$= 2\cos^2\theta(\cos^2\theta-3\sin^2\theta)$
$= 2\cos^2\theta(1-4\sin^2\theta)$

θ	$0 \cdots \frac{\pi}{6} \cdots \frac{\pi}{2}$
$\frac{dx^2}{d\theta}$	$+ \quad 0 \quad -$
x^2	$\nearrow \quad \searrow$

x^2, つまり x が最大になる点Mの偏角は $\theta_0 = \frac{\pi}{6}$ //

(5)



求める面積は, $\theta=\frac{\pi}{6}$ のときの x を x_0 とおく.

$\int_0^{x_0} y\,dx$

$= \int_0^{\frac{\pi}{6}} r\sin\theta \frac{dx}{d\theta} d\theta$

$= \int_0^{\frac{\pi}{6}} r\sin\theta \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right) d\theta$

ここで $r^2=\sin 2\theta$ を θ で微分すると

$2r\frac{dr}{d\theta} = 2\cos 2\theta$

$= \int_0^{\frac{\pi}{6}} (\underbrace{\cos 2\theta}_{1-2\sin^2\theta}\sin\theta\cos\theta - \underbrace{\sin 2\theta}_{2\sin\theta\cos\theta}\sin^2\theta)\,d\theta$

$= \int_0^{\frac{\pi}{6}} (\sin\theta - 2\sin^3\theta - 2\sin^3\theta)\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}} (\sin\theta - 4\sin^3\theta)\cos\theta\,d\theta$

$= \left[\frac{1}{2}\sin^2\theta - \sin^4\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{8} - \frac{1}{16}$

$= \frac{1}{16}$ //

(6) 扇形積分で出す. 求める面積は

$2\int_0^{\frac{\pi}{2}} \frac{1}{2} r^2\,d\theta$

$= \int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta$

$= \left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}$

$= 1$ //