2018 国際医療福祉大

第1問

(A)

(左辺)

$=(x-1)\underbrace{\{x^2+(4a-5)x-(4a-1)\}}_{g(x)}$

$g(x)=0$ において

$D=(4a-5)^2+4(4a-1)$

$=16a^2-24a+21$

$=16\left(a-\frac{3}{4}\right)^2+12>0$

2解を α,β とおく $(\alpha<\beta)$

$\begin{cases}\alpha+\beta=-4a+5\\ \alpha\beta=-4a+1\end{cases}$

$\Downarrow$

$\alpha\beta+4=\alpha+\beta$

$\Leftrightarrow \alpha\beta-\alpha-\beta+1+3=0$

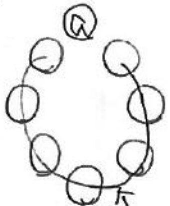
$\Leftrightarrow (\alpha-1)(\beta-1)=-3$

$(\alpha-1,\beta-1)=(-1,3),(-3,1)$

$\therefore (\alpha,\beta)=(0,4),(-2,2)$

$\therefore a=\frac{1}{4},\frac{5}{4}$ //

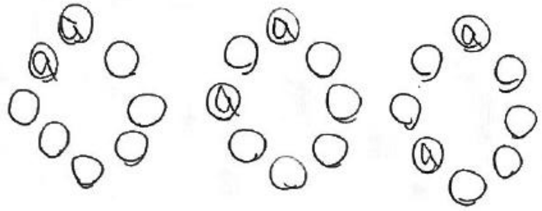
(B)



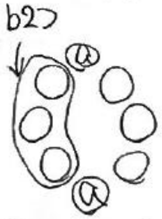
$\frac{7!}{3!4!}=35$ //

(i) aを除くとき　35とおり

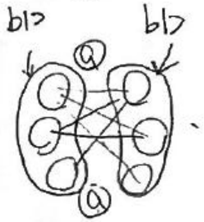
(ii) bを除くとき



$\frac{6!}{2!4!}\times3=45$ とおり

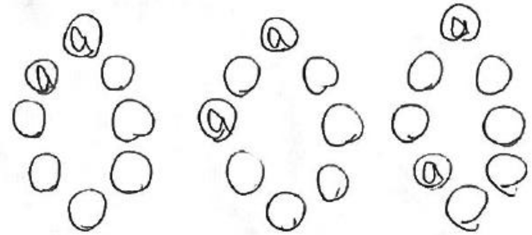


… 3とおり

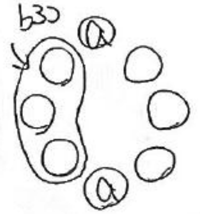


… 6とおり

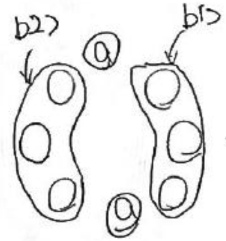
(iii) cを除くとき



$\frac{6!}{3!3!}\times3=60$ とおり



… 1とおり



… ${}_3C_2\cdot{}_3C_1=9$ とおり

以上より

$(i)+(ii)+(iii)=35+54+70$

$=159$ //

(C)

$\ell: x=-2t^2+(6y+2)t-3y$

$=-2\left(t-\frac{3y+1}{2}\right)^2+\frac{9}{2}y^2+\frac{1}{2}$

$=f(t)$　　$(0\leqq t\leqq1)$

yを固定して考える。

(i) $\frac{3y+1}{2}\geqq1\Leftrightarrow y\geqq\frac{1}{3}$ のとき

$f(0)\leqq x\leqq f(1)$

$\Leftrightarrow -3y\leqq x\leqq3y$

(ii) $\frac{1}{2}\leqq\frac{3y+1}{2}\leqq1\Leftrightarrow 0\leqq y\leqq\frac{1}{3}$ のとき

$\underbrace{f(0)}_{-3y}\leqq x\leqq\frac{9}{2}y^2+\frac{1}{2}$

(iii) $0\leqq\frac{3y+1}{2}\leqq\frac{1}{2}\Leftrightarrow -\frac{1}{3}\leqq y\leqq0$ のとき

$\underbrace{f(1)}_{3y}\leqq x\leqq\frac{9}{2}y^2+\frac{1}{2}$

(iv) $\frac{3y+1}{2}\leqq0\Leftrightarrow y\leqq-\frac{1}{3}$ のとき

$f(1)\leqq x\leqq f(0)$

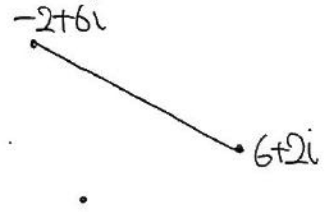
$\Leftrightarrow 3y\leqq x\leqq-3y$

以上より

$|y|\leqq\frac{1}{3}$ のとき　$-3|y|\leqq x\leqq\frac{9}{2}y^2+\frac{1}{2}$ //

$\frac{1}{3}\leqq|y|$ のとき　$-3|y|\leqq x\leqq3|y|$ //

(D)



xy平面上では

$y=-\frac{1}{2}(x-6)+2$

$=-\frac{1}{2}x+5 \quad (-2 \leqq x \leqq 6)$

$z=t+\left(-\frac{1}{2}t+5\right)i \quad (-2 \leqq t \leqq 6)$

とおく.

$|w|^2=w\overline{w}$

$=\frac{1}{z}\cdot\frac{1}{\overline{z}}$

$=\frac{1}{|z|^2}$

$=\frac{1}{\frac{5}{4}t^2-5t+25}$

$=\frac{1}{\frac{5}{4}(t-2)^2+20}$

$t=2$のとき $|w|$も最大. このとき

$w=\frac{1}{2+4i}=\frac{2-4i}{20}$

$=\frac{1-2i}{10}$

$z=x+yi \quad \left(y=-\frac{1}{2}x+5\right)$とおく.

$w=\frac{1}{z}$

$=\frac{1}{x+yi}\cdot\frac{x-yi}{x-yi}$

$=\frac{x-yi}{x^2+y^2}$

$=X+Yi$ とすると

$X=\frac{x}{x^2+y^2} \quad Y=\frac{-y}{x^2+y^2}$

$\frac{dX}{dx}=\frac{x^2+y^2-x(2x+2yy')}{(x^2+y^2)^2}$

$=\frac{x^2+y^2-x(2x-y)}{(x^2+y^2)^2}$

$=\frac{-x^2+y^2+xy}{(x^2+y^2)^2}$

$\frac{dY}{dx}=\frac{-y'(x^2+y^2)+y(2x-y)}{(x^2+y^2)^2}$

$=\frac{x^2-y^2+4xy}{2(x^2+y^2)^2}$

$L=\int_{-2}^{6}\sqrt{\left(\frac{dX}{dx}\right)^2+\left(\frac{dY}{dx}\right)^2}\,dx$

$=\int_{-2}^{6}\sqrt{\frac{4(-x^2+y^2+xy)^2+(x^2-y^2+4xy)^2}{4(x^2+y^2)^4}}\,dx$

$=\int_{-2}^{6}\sqrt{\frac{5(x^2-y^2)^2+20x^2y^2}{4(x^2+y^2)^4}}\,dx$

$=\int_{-2}^{6}\sqrt{\frac{5(x^2+y^2)^2}{4(x^2+y^2)^4}}\,dx$

$=\frac{\sqrt{5}}{2}\int_{-2}^{6}\frac{1}{x^2+y^2}\,dx$

$=\frac{\sqrt{5}}{2}\int_{-2}^{6}\frac{1}{\frac{5}{4}x^2-5x+25}\,dx$

$=\frac{\sqrt{5}}{2}\cdot\frac{4}{5}\int_{-2}^{6}\frac{1}{x^2-4x+20}\,dx$

$=\frac{2\sqrt{5}}{5}\int_{-2}^{6}\frac{1}{(x-2)^2+16}\,dx$

$x-2=4\tan\theta$

$dx=\frac{4}{\cos^2\theta}d\theta$

$=\frac{2\sqrt{5}}{5}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{16(\tan^2\theta+1)}\frac{4}{\cos^2\theta}\,d\theta$

$=\frac{2\sqrt{5}}{5}\left[\frac{1}{4}\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$=\frac{\sqrt{5}}{20}\pi$ //

第2問

$a_{n+1}+\alpha(n+1)+\beta=\frac{1}{2}(a_n+\alpha n+\beta)$

$\Leftrightarrow a_{n+1}=\frac{1}{2}a_n\underbrace{-\frac{1}{2}\alpha n}_{1}\underbrace{-\alpha-\frac{1}{2}\beta}_{\frac{9}{2}}$

$\therefore \alpha=-2,\ \beta=-5$

↓一般項

$a_n-2n-5=(a_1-2-5)\left(\frac{1}{2}\right)^{n-1}$

$=-\frac{1}{8}\left(\frac{1}{2}\right)^{n-1}$

$\therefore a_n=\underline{2n+5-2^{-n-2}}$ //

$b_n=2n+5,\ c_n=\left(\frac{1}{2}\right)^{n+2}$

(1)

$\sum_{k=1}^{n}b_kc_k=S$とおく.

$S=7\cdot\frac{1}{2^3}+9\frac{1}{2^4}+\cdots+(2n+5)\frac{1}{2^{n+2}}$

$\frac{1}{2}S=\qquad 7\frac{1}{2^4}+\cdots+(2n+3)\frac{1}{2^{n+2}}$

$+(2n+5)\frac{1}{2^{n+3}}$

$-)$

$\frac{1}{2}S=\frac{7}{8}+2\frac{1}{2^4}+\cdots+2\frac{1}{2^{n+2}}-\frac{2n+5}{2^{n+3}}$

$=\frac{7}{8}+\left(\frac{1}{8}+\cdots+\frac{1}{2^{n+1}}\right)-\frac{2n+5}{2^{n+3}}$

$S=\frac{7}{4}+\left(\frac{1}{4}+\cdots+\frac{1}{2^n}\right)-\frac{2n+5}{2^{n+2}}$

$=\frac{7}{4}+\frac{\frac{1}{4}-\frac{1}{2^{n+1}}}{1-\frac{1}{2}}-\frac{2n+5}{2^{n+2}}$

$=\frac{7}{4}+\frac{1}{2}-\frac{1}{2^n}-\frac{2n+5}{2^{n+2}}$

$=\frac{9}{4}-(2n+9)2^{-n-2}$

(2)

$\underset{7\ \frac{1}{8}}{b_1c_1}d_1=\frac{1}{6}3\cdot5\cdot7$

$\therefore d_1=\underline{20}$ //

$n \geqq 2$のとき

$b_n c_n d_n$

$= \frac{1}{6}(2n+1)(2n+3)(2n+5)$

$\quad - \frac{1}{6}(2n-1)(2n+1)(2n+3)$

$= (2n+1)(2n+3)$

$\therefore d_n$

$= \frac{(2n+1)(2n+3)}{2n+5} 2^{n+2}$

$= \frac{4n^2+8n+3}{2n+5} 2^{n+2}$ //

$\frac{1}{d_n} = \frac{2n+5}{(2n+1)(2n+3)} \cdot \left(\frac{1}{2}\right)^{n+2}$

$= \left(\frac{2}{2n+1} - \frac{1}{2n+3}\right)\left(\frac{1}{2}\right)^{n+2}$

$= \frac{1}{2n+1}\left(\frac{1}{2}\right)^{n+1} - \frac{1}{2n+3}\left(\frac{1}{2}\right)^{n+2}$

$\sum_{k=1}^{n} \frac{1}{d_k}$

$= \frac{1}{d_1} + \sum_{k=2}^{n} \frac{1}{d_k}$

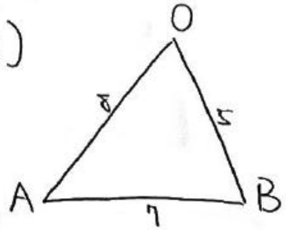
$= \frac{1}{20} + \sum_{k=2}^{n} \left\{ \frac{1}{2k+1}\left(\frac{1}{2}\right)^{k+1} - \frac{1}{2k+3}\left(\frac{1}{2}\right)^{k+2} \right\}$

$= \frac{1}{20} + \frac{1}{5} \cdot \frac{1}{8} - \frac{1}{2n+3}\left(\frac{1}{2}\right)^{n+2}$

$= \frac{3}{40} - \frac{1}{2n+3} 2^{-n-2}$ //

第3問

(1)



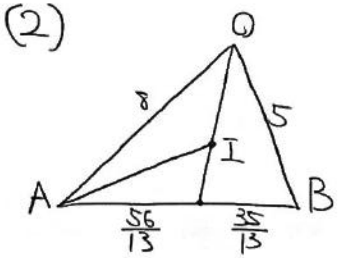
㊀

$\cos\angle AOB = \frac{64+25-49}{2 \cdot 8 \cdot 5}$

$= \frac{1}{2}$

$\therefore \angle AOB = 60°$

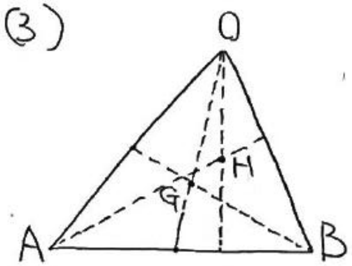
(2)



$\overrightarrow{OI} = \frac{8}{8+\frac{56}{13}} \cdot \frac{5\vec{a}+8\vec{b}}{13}$

$= \frac{5\vec{a}+8\vec{b}}{20} = \frac{1}{4}\vec{a} + \frac{2}{5}\vec{b}$ //

(3)



$\overrightarrow{OH} = s\vec{a} + t\vec{b}$とおく

$\downarrow$

$\overrightarrow{AH} \cdot \vec{b} = \{(s-1)\vec{a} + t\vec{b}\} \cdot \vec{b}$

$= (s-1)20 + 25t = 0$

$\overrightarrow{BH} \cdot \vec{a} = \{s\vec{a} + (t-1)\vec{b}\} \cdot \vec{a}$

$= 64s + (t-1)20 = 0$

$\begin{cases} 4s+5t=4 \\ 16s+5t=5 \end{cases}$

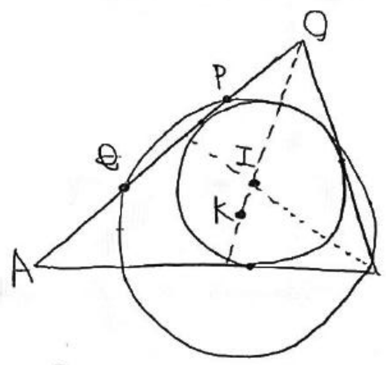
$\therefore s = \frac{1}{12} \quad t = \frac{11}{15}$

$\therefore \overrightarrow{OH} = \frac{1}{12}\vec{a} + \frac{11}{15}\vec{b}$

$\overrightarrow{OK} = \frac{3}{4}\overrightarrow{OG} + \frac{1}{4}\overrightarrow{OH}$

$= \frac{\vec{a}+\vec{b}}{4} + \frac{1}{48}\vec{a} + \frac{11}{60}\vec{b}$

$= \frac{13}{48}\vec{a} + \frac{13}{30}\vec{b}$ //



$\overrightarrow{IK} = \overrightarrow{OK} - \overrightarrow{OI}$

$= \frac{1}{48}\vec{a} + \frac{1}{30}\vec{b}$

$|\overrightarrow{IK}|^2$

$= \left|\frac{1}{48}\vec{a} + \frac{1}{30}\vec{b}\right|^2$

$= \frac{64}{48^2} + \frac{2}{48 \cdot 30} \cdot 20 + \frac{25}{900}$

$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{12}$

$\therefore IK = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$

$\triangle OAB$の内接円の半径

$= \frac{2S}{a+b+c}$

$= \frac{2 \cdot \frac{1}{2} \cdot 5 \cdot 8 \cdot \frac{\sqrt{3}}{2}}{20} = \sqrt{3}$

$\therefore r = \sqrt{3} + \frac{\sqrt{3}}{6} = \frac{7\sqrt{3}}{6}$ //

$\overrightarrow{OP}, \overrightarrow{OQ}$を$u\vec{a}$とおく.

$\overrightarrow{KP} = \left(u - \frac{13}{48}\right)\vec{a} - \frac{13}{30}\vec{b}$

$|\overrightarrow{KP}|^2$

$= \left(u - \frac{13}{48}\right)^2 64 - \frac{13}{15}\left(u - \frac{13}{48}\right)20$

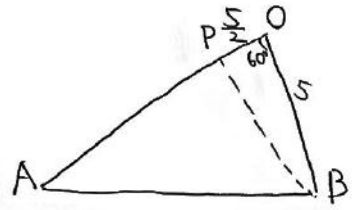
$\quad + \frac{169}{900}25$

$= 64u^2 - 52u + \frac{169}{12} = r^2 = \frac{147}{36}$

$\Leftrightarrow 32u^2 - 26u + 5 = 0$

$\therefore u = \frac{5}{16}, \frac{1}{2}$

$\therefore \overrightarrow{OP} = \frac{5}{16}\vec{a} \quad \overrightarrow{OQ} = \frac{1}{2}\vec{a}$ //



$\angle OPB = \underline{90^\circ}$ //

第4問.

$f(x) = \sqrt{x^2+1} - \frac{1}{2}x - \frac{1}{2}$

$f'(x) = \frac{x}{\sqrt{x^2+1}} - \frac{1}{2}$

$= \frac{2x - \sqrt{x^2+1}}{2\sqrt{x^2+1}}$

(1) $f'(x) = 0 \Leftrightarrow 2x = \sqrt{x^2+1}$

$\downarrow$

$4x^2 = x^2+1$

$\therefore x = \frac{\sqrt{3}}{3} \quad (x \geqq 0)$

x	$0 \cdots \frac{\sqrt{3}}{3} \cdots$
$f'(x)$	$-$ $\quad$ $+$
$f(x)$	$\frac{1}{2} \searrow$ $\quad$ $\nearrow$

$\underline{x = \frac{\sqrt{3}}{3}}$ // で極小値 $\frac{2}{\sqrt{3}} - \frac{\sqrt{3}}{6} - \frac{3}{6}$

$= \underline{\frac{\sqrt{3}-1}{2}}$ //

(2) 接点のx座標をtとおく.

$l: y = \left(\frac{t}{\sqrt{t^2+1}} - \frac{1}{2}\right)(x-t)$

$+ \sqrt{t^2+1} - \frac{1}{2}t - \frac{1}{2}$

$= \left(\frac{t}{\sqrt{t^2+1}} - \frac{1}{2}\right)x + \frac{1}{\sqrt{t^2+1}} - \frac{1}{2}$

$\downarrow$ 原点通る

$0 = \frac{1}{\sqrt{t^2+1}} - \frac{1}{2}$

$\Leftrightarrow t^2+1 = 4 \quad \therefore t = \sqrt{3}$

接点 $\underline{\left(\sqrt{3}, \frac{3-\sqrt{3}}{2}\right)}$ //

(3)

$x = \sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$ とおく

$dx = \cosh\theta \, d\theta$

x	$0 \to \sqrt{3}$
θ	$0 \to \alpha$

$\cosh^2\theta - \sinh^2\theta = 1$

$\int_0^{\sqrt{3}} \sqrt{x^2+1}\, dx$

$= \int_0^{\alpha} \underbrace{\sqrt{\sinh^2\theta + 1}}_{\cosh\theta} \cosh\theta \, d\theta$

$= \int_0^{\alpha} \cosh^2\theta \, d\theta$

$= \int_0^{\alpha} \frac{e^{2\theta} + 2 + e^{-2\theta}}{4} d\theta$

$= \left[\frac{1}{8}e^{2\theta} - \frac{1}{8}e^{2\theta} + \frac{1}{2}\theta\right]_0^{\alpha}$

$= \frac{1}{8}e^{2\alpha} - \frac{1}{8}e^{-2\alpha} + \frac{1}{2}\alpha$

$\frac{e^{\alpha} - e^{-\alpha}}{2} = \sqrt{3}$

$\Leftrightarrow e^{2\alpha} - 2\sqrt{3}e^{\alpha} - 1 = 0$

$\Leftrightarrow e^{\alpha} = \sqrt{3} + 2 \quad (\because e^{\alpha} > 0)$

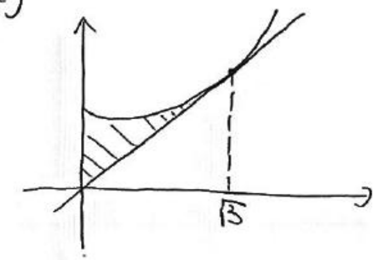
$\Leftrightarrow \alpha = \log(\sqrt{3}+2)$

また $e^{-\alpha} = \frac{1}{\sqrt{3}+2} = 2 - \sqrt{3}$

$= \frac{1}{8}\underbrace{(e^{\alpha} + e^{-\alpha})}_{4}\underbrace{(e^{\alpha} - e^{-\alpha})}_{2\sqrt{3}} + \frac{1}{2}\alpha$

$= \underline{\sqrt{3} + \frac{1}{2}\log(2+\sqrt{3})}$ //

(4)



V

$= \int_0^{\sqrt{3}} \pi \left\{\sqrt{x^2+1} - \frac{1}{2}(x+1)\right\}^2 dx - $ 円錐

$= \pi \int_0^{\sqrt{3}} \left(\frac{5}{4}x^2 + \frac{1}{2}x + \frac{5}{4} - x\sqrt{x^2+1}\right.$

$\left.- \sqrt{x^2+1}\right) dx - $ 円錐

$= \pi \left[\frac{5}{12}x^3 + \frac{1}{4}x^2 + \frac{5}{4}x - \frac{1}{3}(x^2+1)^{\frac{3}{2}}\right]_0^{\sqrt{3}}$

$- \pi \times (3) - $ 円錐

$= \pi\left(\frac{5}{4}\sqrt{3} + \frac{3}{4} + \frac{5}{4}\sqrt{3} - \frac{7}{3}\right)$

$- \pi \times (3) - $ 円錐

$= \pi\left(\frac{3}{2}\sqrt{3} + \frac{9}{12} - \frac{28}{12} - \frac{1}{2}\log(2+\sqrt{3})\right)$

$- \left(\frac{3-\sqrt{3}}{2}\right)^2 \pi \cdot \sqrt{3} \cdot \frac{1}{3}$

$= \pi\left\{\frac{3}{2}\sqrt{3} - \frac{1}{2}\log(2+\sqrt{3}) - \frac{19}{12}\right\}$

$- \frac{12 - 6\sqrt{3}}{4}\pi \cdot \frac{\sqrt{3}}{3}$

$= \underline{\left\{\frac{\sqrt{3}}{2} - \frac{1}{2}\log(2+\sqrt{3}) - \frac{1}{12}\pi\right\}\pi}$ //