2018 慶應理工

1.

(1)

$x \neq 0$ より x^2 で割る

$x^2-2x+3-\frac{2}{x}+\frac{1}{x^2}=0$

$\Leftrightarrow (x+\frac{1}{x})^2-2-2(x+\frac{1}{x})+3=0$

$\Leftrightarrow \underline{y^2-2y+1=0}$

$\Leftrightarrow y=x+\frac{1}{x}=1$

$\Leftrightarrow x^2-x+1=0 \quad (x\neq 0)$

$\Leftrightarrow x=\frac{1\pm\sqrt{3}i}{2}$

(2)

x^3+y^3+xy-3

$=(x+y)^3-3xy(x+y)+xy-3$

$=s^3-3ts+t-3=0$

$\Leftrightarrow (1-3s)t=-s^3+3$

$\therefore \quad t=\frac{s^3-3}{3s-1}$

また x, y を解にもつ2次方程式は

$u^2-su+t=0$

これの実数解が存在する条件は

$D=s^2-4t\geqq 0$

$\Leftrightarrow t\leqq \frac{1}{4}s^2$

$\Downarrow$

$\frac{s^3-3}{3s-1}\leqq \frac{1}{4}s^2$

(i) $3s-1>0 \Leftrightarrow s>\frac{1}{3}$ のとき

$4s^3-12\leqq 3s^3-s^2$

$\Leftrightarrow s^3+s^2-12\leqq 0$

$\Leftrightarrow (s-2)\underbrace{(s^2+3s+6)}_{>0}\leqq 0$

$\Leftrightarrow s-2\leqq 0$

$\therefore s\leqq 2$

(ii) $3s-1<0 \Leftrightarrow s<\frac{1}{3}$ のとき

$4s^3-12\geqq 3s^3-s^2$

$\Leftrightarrow (s-2)\underbrace{(s^2+3s+6)}_{>0}\geqq 0$

$\Leftrightarrow s\geqq 2$

これは $s<\frac{1}{3}$ を満たさない。

(i)(ii)より

$\underline{\frac{1}{3}<s\leqq 2}$

(3)

$(x-1)(x^{3n}-1)$

$=(x-1)(x^n-1)(x^{2n}+x^n+1)$

$\Downarrow$

$x^{2n}+x^n+1=(x^2+x+1)Q(x) \quad \cdots①$

と表せることを示す。

$x^{2n}+x^n+1=x^{6m+2}+x^{3m+1}+1=f(x)$

$x^2+x+1=g(x) \qquad (m\in\mathbb{Z})$

とおく。

$\omega=\frac{-1+\sqrt{3}i}{2}$

$\omega^2=\frac{-1-\sqrt{3}i}{2}$ とおく。

$g(\omega)=\omega^2+\omega+1=0$

$g(\omega^2)=\omega^4+\omega^2+1$

$=\omega+\omega^2+1=0$

$\therefore g(x)=(x-\omega)(x-\omega^2)$

と表せる。また

$f(\omega)=\omega^{6m+2}+\omega^{3m+1}+1$

$=(\omega^3)^{2m}\omega^2+(\omega^3)^m\omega+1$

$=\omega^2+\omega+1=0$

$f(\omega^2)=\omega^{12m+4}+\omega^{6m+2}+1$

$=(\omega^3)^{4m+1}\omega+(\omega^3)^{2m}\omega^2+1$

$=\omega+\omega^2+1=0$

より $f(x)$ も $(x-\omega)$ と $(x-\omega^2)$ を因数にもつので $f(x)$ は $g(x)$ で割り切れる。①が成立つので

$(x-1)(x^{3n}-1)$

$=(x-1)(x^n-1)(x^2+x+1)Q(x)$

$=(x^3-1)(x^n-1)Q(x)$

題意は示された。

2.

(1)

$P(\text{y座標が}0)$

$= P(\rightarrow \times 2) + P(\uparrow \times 1, \downarrow \times 1)$

$= \left(\frac{1}{3}\right)^2 + {}_2C_1\left(\frac{1}{3}\right)^2 = \underline{\frac{1}{3}}$ //

(2)

$P(3\text{回で}x\text{または}y\text{が}1\text{以上})$

$= 1 - P(3\text{回で}x\text{かつ}y\text{が}1\text{未満})$

$= 1 - P(\downarrow \times 3) - P(\downarrow \times 2, \uparrow \times 1)$

$= 1 - \left(\frac{1}{3}\right)^3 - {}_3C_1\left(\frac{1}{3}\right)^3$

$= \underline{\frac{23}{27}}$ //

(3)

$P(4\text{回で}x\text{が}2\text{以上})$

$= 1 - P(4\text{回で}x\text{が}0) - P(4\text{回で}x\text{が}1)$

$= 1 - \left(\frac{2}{3}\right)^4 - {}_4C_1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3$

$= \frac{81 - 16 - 32}{81}$

$= \frac{33}{81} = \underline{\frac{11}{27}}$ //

$P_{4\text{回で}x\text{が}2\text{以上}}(y\text{が}0)$

$= \dfrac{P(4\text{回で}x\text{が}2\text{以上かつ}y\text{が}0)}{\frac{11}{27}}$

$= \frac{27}{11}\{P(\rightarrow \times 2, \uparrow \times 1, \downarrow \times 1)$

$+ P(\rightarrow \times 4)\}$

$= \frac{27}{11}\left\{\frac{4!}{2!1!1!}\left(\frac{1}{3}\right)^2 \frac{1}{3}\cdot\frac{1}{3} + \left(\frac{1}{3}\right)^4\right\}$

$= \frac{27}{11}\left(\frac{12}{81} + \frac{1}{81}\right) = \underline{\frac{13}{33}}$ //

(4)

$P(n\text{回ではじめて}x\text{が}2)$

$= P(\sim\sim \rightarrow \sim\sim \rightarrow)$

$= {}_{n-1}C_1 \frac{1}{3}\left(\frac{2}{3}\right)^{n-2} \cdot \frac{1}{3}$

$= \underline{\frac{(n-1)2^{n-2}}{3^n}}$ //

(5)

k回目　k+1回目

$P(\underbrace{\sim\sim}_{\rightarrow \times 1} \rightarrow \underset{\rightarrow \text{リスタート}}{\sim} \sim\sim)$

$= \sum_{k=2}^{n-1} {}_{k-1}C_1 \frac{1}{3}\left(\frac{2}{3}\right)^{k-2} \cdot \frac{1}{3} \cdot \frac{2}{3}$

$= \frac{2}{27} \underbrace{\sum_{k=2}^{n-1} (k-1)\left(\frac{2}{3}\right)^{k-2}}_{S(n)}$

$S(n) = 1 + 2\cdot\frac{2}{3} + 3\left(\frac{2}{3}\right)^2 + \cdots + (n-2)\left(\frac{2}{3}\right)^{n-3}$

$-\frac{2}{3}S(n) = \quad 1\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + \cdots + (n-3)\left(\frac{2}{3}\right)^{n-3} + (n-2)\left(\frac{2}{3}\right)^{n-2}$

$\frac{1}{3}S(n) = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{n-3} - (n-2)\left(\frac{2}{3}\right)^{n-2}$

$= \frac{1 - \left(\frac{2}{3}\right)^{n-2}}{1 - \frac{2}{3}} - (n-2)\left(\frac{2}{3}\right)^{n-2}$

$= 3 - 3\left(\frac{2}{3}\right)^{n-2} - (n-2)\left(\frac{2}{3}\right)^{n-2}$

$= 3 - (n+1)\left(\frac{2}{3}\right)^{n-2}$

$\therefore S(n) = 9 - 3(n+1)\left(\frac{2}{3}\right)^{n-2}$

求める確率は

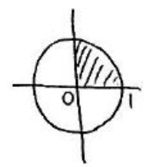
$\underline{\frac{2}{27}S(n) = \frac{2}{3} - \frac{2}{9}(n+1)\left(\frac{2}{3}\right)^{n-2}}$ //

3.

(1) $a_1 = \int_0^1 (1 - x^2)^{\frac{1}{2}} dx$

$= \int_0^1 \sqrt{1 - x^2}\, dx$

$= \frac{\pi}{4}$



a_n

$= \int_0^1 (x)'(1 - x^2)^{\frac{n}{2}} dx$

$= \left[x(1 - x^2)^{\frac{n}{2}}\right]_0^1$

$- \int_0^1 x(-2x)\frac{n}{2}(1 - x^2)^{\frac{n-2}{2}} dx$

$= n\int_0^1 \underbrace{x^2}_{x^2 - 1 + 1}(1 - x^2)^{\frac{n-2}{2}} dx$

$= -na_n + na_{n-2}$

$\Leftrightarrow \underline{a_n = \frac{n}{1+n}a_{n-2}}$ //

(2)

$a_n a_{n-1} = \frac{n}{1+n} \cdot \frac{n-1}{n} a_{n-2} a_{n-3}$

$= \frac{n-1}{1+n} a_{n-2} a_{n-3}$

$\downarrow$

$a_n a_{n-1} = \frac{n-1}{1+n} \cdot \frac{n-3}{n-1} \cdot \frac{n-5}{n-3} \cdots \frac{3}{5} a_2 a_1$

$= \frac{3}{1+n} \cdot \frac{2}{3} \cdot \frac{\pi}{4}$

$= \underline{\frac{\pi}{2(n+1)}}$ //

(3)

$0 \leqq x \leqq 1$ において

$0 \leqq 1-x^2 \leqq 1$

より

$0 \leqq (1-x^2)^{\frac{n}{2}} \leqq (1-x^2)^{\frac{n-1}{2}}$

から この範囲で積分して

$0 \leqq a_n \leqq a_{n-1}$.

これより

$\frac{a_{n+1}}{a_{n-1}} \leqq \frac{a_n}{a_{n-1}} \leqq \frac{a_{n-1}}{a_{n-1}}$

$\Leftrightarrow \frac{n+1}{n+2} \leqq \frac{a_n}{a_{n-1}} \leqq 1$

$\lim_{n\to\infty} \frac{n+1}{n+2} = 1$ より 題意が示された.

(4)

$\lim_{n\to\infty} \sqrt{n}\, a_n$

$= \lim_{n\to\infty} \sqrt{n a_n^2}$

$= \lim_{n\to\infty} \sqrt{n a_n a_{n-1} \cdot \frac{a_n}{a_{n-1}}}$

$= \lim_{n\to\infty} \sqrt{\frac{\pi}{2(\frac{1}{n}+1)} \cdot \frac{a_n}{a_{n-1}}}$

$= \sqrt{\frac{\pi}{2}}$ //

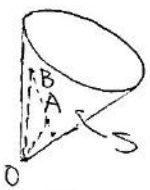
4.

(1)

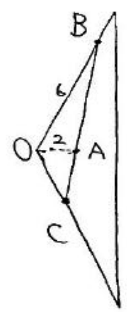
$|\overrightarrow{OP}| = \overrightarrow{OA} \cdot \overrightarrow{OP}$

$= 2|\overrightarrow{OP}| \cos\angle AOP$

$\therefore \angle AOP = \frac{\pi}{3}$



$\angle AOB = \frac{\pi}{3}$ //



△OBCの面積に関して

$\frac{1}{2} OC \cdot 2 \sin\frac{\pi}{3} + \frac{1}{2} \cdot 6 \cdot 2 \sin\frac{\pi}{3}$

$= \frac{1}{2} OC \cdot 6 \cdot \sin\frac{2}{3}\pi$

$\Leftrightarrow 2OC + 12 = 6OC$

$\therefore OC = 3$

$\therefore CA : AB = 1 : 2$

$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$

$= \overrightarrow{OB} + \frac{3}{2}\overrightarrow{BA}$

$= \frac{3}{2}\overrightarrow{OA} - \frac{1}{2}\overrightarrow{OB}$ //

$\overrightarrow{OC} \cdot \overrightarrow{OD}$

$= (\frac{3}{2}\overrightarrow{OA} - \frac{1}{2}\overrightarrow{OB}) \cdot \overrightarrow{OD}$

$= \frac{3}{2}|\overrightarrow{OD}| - \frac{1}{2}\overrightarrow{OB} \cdot \overrightarrow{OD}$

DはPと同じ

$= \frac{3}{2}\underbrace{|\overrightarrow{OC}|}_{3} - \frac{1}{2}\overrightarrow{OB} \cdot \overrightarrow{OD} = 0$

$\therefore \overrightarrow{OB} \cdot \overrightarrow{OD} = 9$ //

(2)

まずQは平面A,B,D上.

$s+t+u=1$,

また

$|\overrightarrow{OQ}|^2$

$= 4s^2 + 36t^2 + 9u^2 + 2st \cdot 6$
$\quad + 2tu \cdot 9 + 2us \cdot 3$

$\overrightarrow{OQ} \cdot \overrightarrow{OA}$

$= 4s + 6t + 3u = |\overrightarrow{OQ}|$

$\Downarrow$

$4s^2 + 36t^2 + 9u^2 + 12st + 18tu + 6us$

$= (4s + 6t + 3u)^2$

$\Leftrightarrow 0 = 12s^2 + 36st + 18tu + 18us$

$\Leftrightarrow 0 = 2s^2 + 6st + 3t(1-s-t)$
$\quad + 3(1-s-t)s$

$\Leftrightarrow 0 = -s^2 - 3t^2 + 3s + 3t$

$\therefore s^2 + 3t^2 - 3s - 3t = 0$ //

(3)

$|\overrightarrow{OQ}|$

$= 4s + 6t + 3(1-s-t)$

$= s + 3t + 3$

$= k + 3$ とおく. ($s + 3t = k$)

(2)より

$(k-3t)^2 + 3t^2 - 3(k-3t) - 3t = 0$

$\Leftrightarrow 12t^2 + (6-6k)t + k^2 - 3k = 0$

tは実数解をもつので

$\frac{D}{4} = (3-3k)^2 - 12(k^2 - 3k)$

$= -3k^2 + 18k + 9 \geqq 0$

$\Leftrightarrow k^2 - 6k - 3 \leqq 0$

$\Leftrightarrow 3 - 2\sqrt{3} \leqq k \leqq 3 + 2\sqrt{3}$

$k = 3 + 2\sqrt{3}$ で最大. このとき

$12t^2 + (-12 - 12\sqrt{3})t + 12 + 6\sqrt{3} = 0$

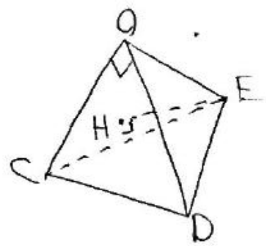
$\Leftrightarrow t = \frac{1+\sqrt{3}}{2}$

$s = 3 + 2\sqrt{3} - \frac{3+3\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2}$

このとき

$\overrightarrow{OE} = \frac{3+\sqrt{3}}{2}\overrightarrow{OA} + \frac{1+\sqrt{3}}{2}\overrightarrow{OB}$

$\quad + (-\sqrt{3}-1)\overrightarrow{OD}$ //



EからΔOCDに垂ろした垂線の足をHとおく.

$\overrightarrow{OH} = \alpha\overrightarrow{OC} + \beta\overrightarrow{OD}$.

$\overrightarrow{EH}$

$= -\frac{3+\sqrt{3}}{2}\overrightarrow{OA} - \frac{1+\sqrt{3}}{2}\overrightarrow{OB}$

$\quad + \alpha\overrightarrow{OC} + (\beta+\sqrt{3}+1)\overrightarrow{OD}$

$\overrightarrow{EH}\cdot\overrightarrow{OC}$

$= -\frac{9+3\sqrt{3}}{2} + \frac{9+9\sqrt{3}}{2} + 9\alpha = 0$

$\therefore \alpha = -\frac{\sqrt{3}}{3}$

$\overrightarrow{EH}\cdot\overrightarrow{OD}$

$= -\frac{9+3\sqrt{3}}{2} - \frac{9+9\sqrt{3}}{2} + 9\beta + 9\sqrt{3} + 9 = 0$

$\therefore \beta = -\frac{\sqrt{3}}{3}$

$\overrightarrow{EH}$

$= -\frac{3+\sqrt{3}}{2}\overrightarrow{OA} - \frac{1+\sqrt{3}}{2}\overrightarrow{OB}$

$\quad - \frac{\sqrt{3}}{3}\left(\frac{3}{2}\overrightarrow{OA} - \frac{1}{2}\overrightarrow{OB}\right)$

$\quad + \left(\frac{2}{3}\sqrt{3}+1\right)\overrightarrow{OD}$

$= \frac{3+2\sqrt{3}}{6}\left(-3\overrightarrow{OA} - \overrightarrow{OB} + 2\overrightarrow{OD}\right)$

$|\overrightarrow{EH}|$

$= \frac{3+2\sqrt{3}}{6}\sqrt{36+36+36+36-36-36}$

$= \frac{3+2\sqrt{3}}{6}6\sqrt{2} = 2\sqrt{6}+3\sqrt{2}$

(四面体OCDE)

$= \frac{9}{2} \times (2\sqrt{6}+3\sqrt{2}) \times \frac{1}{3}$

$= \frac{6\sqrt{6}+9\sqrt{2}}{2}$ //

5.

(1) $OQ : AQ = 1 : v$ 　 $Q(X, Y)$とおく.

↓ 2乗

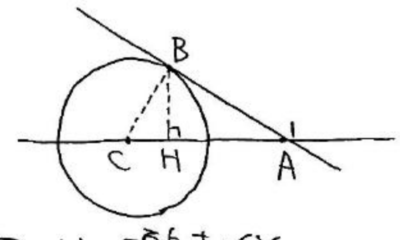
$(X-1)^2 + Y^2 = v^2(X^2+Y^2)$

$\Leftrightarrow (1-v^2)X^2 - 2X + (1-v^2)Y^2 + 1 = 0$

$\Leftrightarrow \left(X - \frac{1}{1-v^2}\right)^2 + Y^2 = \frac{1}{(1-v^2)^2} - \frac{1}{1-v^2}$

$\Leftrightarrow \left(X - \frac{1}{1-v^2}\right)^2 + Y^2 = \left(\frac{v}{1-v^2}\right)^2$

(半径) $= \frac{v}{v^2-1}$ // 　 $(v>1)$



図のように頂点おくと

$\triangle ABC \sim \triangle AHB$.

$AC : CB = 1 - \frac{1}{1-v^2} : \frac{v}{v^2-1}$

$= v : 1 = AB : BH$

これはHとOが等しいことを意味する.

Bのx座標はOより

$B\left(0, \frac{1}{\sqrt{v^2-1}}\right)$ //

(2)

(最短経路) 　 $\downarrow \begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$

$= AB + \int_{\theta_1}^{\theta_2}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$= AB$

$+ \int_{\theta_1}^{\theta_2}\sqrt{\{f'(\theta)\cos\theta - f(\theta)\sin\theta\}^2}$

$\overline{\quad + \{f'(\theta)\sin\theta + f(\theta)\cos\theta\}^2}\,d\theta$

$= AB$

$+ \int_{\theta_0}^{\theta_1}\sqrt{\{f'(\theta)\}^2 + \{f(\theta)\}^2}\,d\theta$ //

$= vf(\theta_1)$

$\downarrow f(\theta) = \beta e^{\alpha(\theta-\theta_0)}$

(i) $\alpha = 0$のとき

$v\beta = \sqrt{1 + \left(\frac{1}{v^2-1}\right)^2} + \int_{\theta_0}^{\theta_1}|\beta|\,d\theta$

$\Leftrightarrow v\beta = \frac{v}{\sqrt{v^2-1}} + |\beta|(\theta_1 - \theta_0)$

どんなθ_1に対しても成り立つには

$\beta = 0$ かつ $v = 0$. これは不適.

(ii) $\alpha \neq 0$のとき

$v\beta e^{\alpha(\theta_1-\theta_0)}$

$= \frac{v}{\sqrt{v^2-1}} + \sqrt{1+\alpha^2}|\beta|\int_{\theta_0}^{\theta_1}e^{\alpha(\theta-\theta_0)}\,d\theta$

$\Leftrightarrow$

$v\beta e^{\alpha(\theta_1-\theta_0)}$

$= \frac{v}{\sqrt{v^2-1}} + \frac{\sqrt{1+\alpha^2}}{\alpha}|\beta|\{e^{\alpha(\theta_1-\theta_0)} - 1\}$

$\theta_1 = \theta_0$のときも成立している必要があるので

$v\beta = \frac{v}{\sqrt{v^2-1}}$

$\therefore \beta = \frac{1}{\sqrt{v^2-1}}$

またどんなθでも成り立つので

$$0=\frac{v}{\sqrt{v^2-1}}-\frac{\sqrt{1+\alpha^2}}{\alpha}|\beta|$$

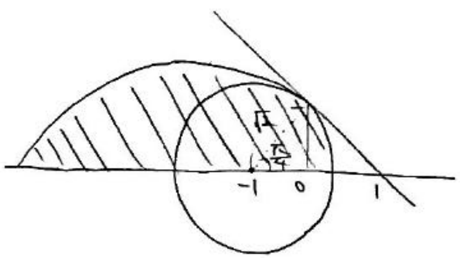
$$\Leftrightarrow 0=v-\frac{\sqrt{1+\alpha^2}}{\alpha}$$

$$\Leftrightarrow \alpha v=\sqrt{1+\alpha^2}$$

2乗して解くと

$$\underline{\alpha=\frac{1}{\sqrt{v^2-1}}}\ /\!/$$

$v=\sqrt{2}$のとき



求める面積は

$2\{\int_{-s(\pi)}^{0} y\,dx +$  $\}$

$$=2\int_{\pi}^{\frac{\pi}{2}} e^{\theta-\frac{\pi}{2}}\sin\theta\{e^{\theta-\frac{\pi}{2}}(\cos\theta-\sin\theta)\}d\theta$$

$+$  $-$ 

$$=2\int_{\frac{\pi}{2}}^{\pi} e^{2\theta-\pi}(\sin^2\theta-\sin\theta\cos\theta)d\theta+\frac{\pi}{2}-1$$

$$=e^{-\pi}\int_{\frac{\pi}{2}}^{\pi} e^{2\theta}(1-\cos2\theta-\sin2\theta)d\theta+\frac{\pi}{2}-1$$

$$=e^{-\pi}\left[\frac{1}{2}e^{2\theta}-\frac{1}{2}e^{2\theta}\sin2\theta\right]_{\frac{\pi}{2}}^{\pi}+\frac{\pi}{2}-1$$

$=\cdots$

$$=\underline{\frac{1}{2}(e^{\pi}+\pi-3)}\ /\!/$$