2018 東京慈恵会 医

1.

(1) Nが3の倍数

組	順列(区別する)
(1, 2, 3)	$2\cdot3\cdot3!$ とり
(1, 4, 4)	${}_4C_2\cdot3!$
(2, 3, 4)	$2\cdot3\cdot4\cdot3!$
(3, 3, 3)	$3!$
(4, 4, 4)	${}_4C_3\cdot3!$

P(Nが3の倍数)

$$=\frac{(6+6+24+1+4)3!}{{}_{10}P_3}$$

$$=\frac{41\cdot6}{10\cdot9\cdot8}=\frac{41}{120}$$ //

Nが6の倍数

組	順列(区別する)
(1, 2, 3)	$2\cdot3\cdot2$ とり
(1, 4, 4)	${}_4C_2\times2\cdot2$
(2, 3, 4)	$2\cdot3\cdot4\times2\cdot2$
(4, 4, 4)	${}_4C_3\cdot3!$

P(Nが6の倍数)

$$=\frac{12+24+96+24}{{}_{10}P_3}=\frac{156}{10\cdot9\cdot8}=\frac{13}{60}$$ //

(2)

$$\begin{cases}2x+y=X\\2x-y=Y\end{cases}\iff x=\frac{X+Y}{4},\ y=\frac{X-Y}{2}$$

K

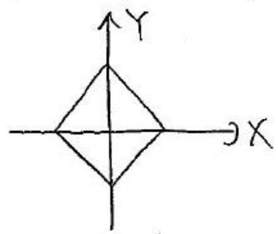
$$=2x^2+xy-y^2$$

$$=2\frac{(X+Y)^2}{16}+\frac{1}{8}(X^2-Y^2)-\frac{1}{4}(X-Y)^2$$

$$=\frac{1}{8}(2X^2+2XY)-\frac{1}{4}(X^2-2XY+Y^2)$$

$$=\frac{3}{4}XY-\frac{1}{4}Y^2$$

$$|2x+y|+|2x-y|=4\iff|X|+|Y|=4$$



min kはXとYが異符号のとき,

つまり $Y=X\pm4$.

$$K=\frac{3}{4}X(X\pm4)-\frac{1}{4}(X\pm4)^2$$

$$=\frac{1}{2}X^2\pm3X-(\pm2X)-4$$

$$=\frac{1}{2}X^2\pm X-4$$

$$=\frac{1}{2}(X\pm1)^2-\frac{9}{2}$$

$(X, Y)=(\mp1, \pm3)$のとき

$$\min k=-\frac{9}{2}.$$

max kはXとYが同符号のとき

つまり $Y=-X\pm4$.

$$K=\frac{3}{4}X(-X\pm4)-\frac{1}{4}(-X\pm4)^2$$

$$=-X^2\pm3X-(\mp2X)-4$$

$$=-X^2\pm5X-4$$

$$=-\left(X\mp\frac{5}{2}\right)^2+\frac{9}{4}$$

$(X, Y)=\left(\pm\frac{5}{2}, \pm\frac{3}{2}\right)$のとき

$$\max k=\frac{9}{4}$$

$$\therefore -\frac{9}{2}\leqq2x^2+xy-y^2\leqq\frac{9}{4}$$ //

2.

(1) $x-t=u$ とおく.

$f_n(x)$

$$=e^{-x}x^{n+1}+\int_x^0e^{u-x}f_n(u)(-du)$$

$$=e^{-x}x^{n+1}+e^{-x}\int_0^xe^uf_n(u)du\cdots①$$

$\frac{d}{dx}f_n(x)$

$$=-e^{-x}x^{n+1}+(n+1)e^{-x}x^n$$

$$-e^{-x}\int_0^xe^uf_n(u)du+e^{-x}e^xf_n(x)$$

①

$$=-e^{-x}x^{n+1}+(n+1)e^{-x}x^n+e^{-x}x^{n+1}$$

$$=(n+1)e^{-x}x^n$$ //

(2)

$g(x)=e^{-x}x^m$ とおく.

$$g'(x)=-e^{-x}x^m+me^{-x}x^{m-1}$$

$$=(m-x)e^{-x}x^{m-1}$$

増減表

x	$0\cdots m\cdots$
$g'(x)$	× + 0 −
$g(x)$	× ↗ ↘

より $g(x)\leqq g(m)$

$$\therefore e^{-x}x^m\leqq e^{-m}m^m$$

(3)

$f_n(x)$

$$=\int(n+1)e^{-x}x^ndx$$

$$=-(n+1)e^{-x}x^n+\int(n+1)ne^{-x}x^{n-1}dx$$

$$=-(n+1)e^{-x}x^n-(n+1)ne^{-x}x^{n-1}$$

$$-\cdots-(n+1)!e^{-x}x+\int(n+1)!e^{-x}dx$$

$$=-(n+1)e^{-x}x^n-(n+1)ne^{-x}x^{n-1}$$

$$-\cdots-(n+1)!e^{-x}x-(n+1)!e^{-x}+C$$

↓ $x=0$

$$f_n(0)=-(n+1)!+C=0$$

$$\therefore C=(n+1)!$$

ここで(2)より，$x>0$ のとき

$$0<e^{-x}x^{n+1}\leqq e^{-(n+1)}(n+1)^{n+1}$$

$$\Leftrightarrow 0<e^{-x}x^{n}\leqq \frac{1}{x}e^{-(n+1)}(n+1)^{n+1}$$

より 最右辺が $x\to\infty$ で 0 に収束するので

$$\lim_{x\to\infty} e^{-x}x^{n}=0$$

つまり

$$\lim_{x\to\infty} e^{-x}x^{N}=0 \quad (N\geqq 1)$$

よって

$$\lim_{x\to\infty} f_n(x)=C$$
$$=\underline{(n+1)!}$$

3.

(1) $f_2(x)$

$$=f_1(f_1(x))$$
$$=\left(x^2+x-\frac{1}{4}\right)^2+x^2+x-\frac{1}{4}-\frac{1}{4}$$
$$=x^2\left\{x^2+2\left(x-\frac{1}{4}\right)+1\right\}$$
$$+\underbrace{\left(x-\frac{1}{4}\right)^2}_{x^2-\frac{1}{2}x+\frac{1}{16}}+x-\frac{1}{4}-\frac{1}{4}$$

$$a_2x+b_2=-\frac{1}{2}x+\frac{1}{16}+x-\frac{1}{2}=\frac{x}{2}-\frac{7}{16}$$

$$\therefore\ \underline{a_2=\frac{1}{2},\ b_2=-\frac{7}{16}}$$

(2)

以下 x^2 を法とする合同式とする.

$$f_{n+1}(x)\equiv\{f_n(x)\}^2+f_n(x)-\frac{1}{4}$$
$$\equiv(a_nx+b_n)^2+a_nx+b_n-\frac{1}{4}$$
$$\equiv(2a_nb_n+a_n)x+b_n^2+b_n-\frac{1}{4}$$

$$\downarrow$$

$$\begin{cases}a_{n+1}=2a_nb_n+a_n=(2b_n+1)a_n\\ b_{n+1}=b_n^2+b_n-\frac{1}{4}\end{cases}$$

$$\downarrow$$

$$b_{n+1}=\left(b_n+\frac{1}{2}\right)^2-\frac{1}{2}$$
$$\Leftrightarrow b_{n+1}+\frac{1}{2}=\left(b_n+\frac{1}{2}\right)^2$$

$b_n+\frac{1}{2}$ $(n=1,2,3,\cdots)$ は正なので対数とると

$$\log_2\left(b_{n+1}+\frac{1}{2}\right)=2\log_2\left(b_n+\frac{1}{2}\right)$$

$$\downarrow \text{一般項}$$

$$\log_2\left(b_n+\frac{1}{2}\right)=\left\{\log_2\left(b_1+\frac{1}{2}\right)\right\}2^{n-1}$$
$$=2^{n-1}\log_2\frac{1}{4}$$

$$\therefore\ b_n+\frac{1}{2}=\left(\frac{1}{4}\right)^{2^{n-1}}$$

これより

$$a_{n+1}=2\left(\frac{1}{4}\right)^{2^{n-1}}a_n\leqq 2\cdot\frac{1}{4}a_n$$

より

$$0<|a_n|\leqq\frac{1}{2}|a_{n-1}|$$
$$\leqq\left(\frac{1}{2}\right)^2|a_{n-2}|$$
$$\leqq\cdots\leqq\left(\frac{1}{2}\right)^{n-1}|a_1|$$

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^{n-1}|a_1|=0 \text{ より}$$

$$\lim_{n\to\infty}|a_n|=0 \quad \therefore\ \underline{\lim_{n\to\infty}a_n=0}$$

4.

(1) $|\overrightarrow{OQ}|=\sqrt{16\cos^2\theta+\frac{1}{2}\sin^2\theta+\frac{1}{2}\sin^2\theta}$

$$=\sqrt{15\cos^2\theta+1}$$

$$\overrightarrow{PR}$$
$$=\overrightarrow{OR}-\overrightarrow{OP}$$
$$=\frac{4}{|\overrightarrow{OQ}|}\begin{pmatrix}4\cos\theta\\ \frac{\sqrt{2}}{2}\sin\theta\\ \frac{\sqrt{2}}{2}\sin\theta\end{pmatrix}-\begin{pmatrix}4\cos\theta\\ 2\sqrt{2}\sin\theta\\ 2\sqrt{2}\sin\theta\end{pmatrix}$$
$$=\begin{pmatrix}\left(\frac{16}{|\overrightarrow{OQ}|}-4\right)\cos\theta\\ \left(\frac{2\sqrt{2}}{|\overrightarrow{OQ}|}-2\sqrt{2}\right)\sin\theta\\ \left(\frac{2\sqrt{2}}{|\overrightarrow{OQ}|}-2\sqrt{2}\right)\sin\theta\end{pmatrix}$$

$$|\overrightarrow{PR}|^2$$
$$=\left(\frac{256}{|\overrightarrow{OQ}|^2}-\frac{128}{|\overrightarrow{OQ}|}+16\right)\cos^2\theta+2\left(\frac{8}{|\overrightarrow{OQ}|^2}-\frac{16}{|\overrightarrow{OQ}|}+8\right)\sin^2\theta$$
$$=\frac{240}{|\overrightarrow{OQ}|^2}\cos^2\theta-\frac{96}{|\overrightarrow{OQ}|}\cos^2\theta+\frac{16}{|\overrightarrow{OQ}|^2}-\frac{32}{|\overrightarrow{OQ}|}+16$$
$$=\frac{16}{|\overrightarrow{OQ}|^2}(15\cos^2\theta+1)-\frac{32}{|\overrightarrow{OQ}|}(3\cos^2\theta+1)+16$$
$$=-\frac{32(3\cos^2\theta+1)}{\sqrt{15\cos^2\theta+1}}+32$$
$$=-\frac{32\left(3\cos^2\theta+\frac{1}{5}\right)+\frac{4}{5}32}{\sqrt{15\cos^2\theta+1}}+32$$

$$\downarrow \sqrt{15\cos^2\theta+1}=u$$

$$=-\frac{\frac{32}{5}u^2+\frac{4}{5}\cdot 32}{u}+32$$
$$=-\frac{32}{5}\left(u+\frac{4}{u}\right)+32$$

$u>0,\ \frac{4}{u}>0$ (相加平均)≧(相乗平均) より

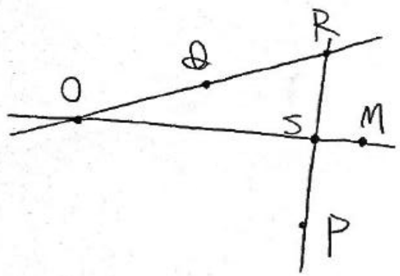
$$\leqq-\frac{32}{5}2\sqrt{u\cdot\frac{4}{u}}+32=\frac{32}{5}$$

$$u=\frac{4}{u}\Leftrightarrow 15\cos^2\theta+1=4$$
$$\Leftrightarrow \cos\theta=\frac{1}{\sqrt{5}}\quad\left(0<\theta<\frac{\pi}{2}\right)$$

のとき $|\overrightarrow{PR}|$ の最大値 $\sqrt{\frac{32}{5}}$

$$=\frac{4\sqrt{2}}{\sqrt{5}}$$
$$=\underline{\frac{4\sqrt{10}}{5}}$$

(2)



$\cos\theta = \frac{1}{\sqrt{5}}$ のとき $\sin\theta = \frac{2}{\sqrt{5}}$,

$|\overrightarrow{OQ}| = 2$. $\overrightarrow{OR} = 2\overrightarrow{OQ}$.

$P\left(\frac{4}{\sqrt{5}}, \frac{4\sqrt{2}}{\sqrt{5}}, \frac{4\sqrt{2}}{\sqrt{5}}\right)$

$R\left(\frac{8}{\sqrt{5}}, \frac{2\sqrt{2}}{\sqrt{5}}, \frac{2\sqrt{2}}{\sqrt{5}}\right)$

PRの中点Sとおくと

$S\left(\frac{6}{\sqrt{5}}, \frac{3\sqrt{2}}{\sqrt{5}}, \frac{3\sqrt{2}}{\sqrt{5}}\right)$

$$|\overrightarrow{OS}| = \sqrt{\frac{36}{5} + \frac{18}{5} + \frac{18}{5}}$$

$$= \frac{6\sqrt{2}}{\sqrt{5}}$$

$$\overrightarrow{OM} = 4 \cdot \frac{\sqrt{5}}{6\sqrt{2}} \overrightarrow{OS}$$

$$= \begin{pmatrix} 2\sqrt{2} \\ 2 \\ 2 \end{pmatrix}$$

$\triangle MOA$

$$= \frac{1}{2}\sqrt{|\overrightarrow{OM}|^2|\overrightarrow{OA}|^2 - (\overrightarrow{OM}\cdot\overrightarrow{OA})^2}$$

$$= \frac{1}{2}\sqrt{\underbrace{16\cdot 16}_{256} - \underbrace{(8\sqrt{2})^2}_{128}}$$

$$= \underline{4\sqrt{2}} \;/\!/$$