北海道大 2018

$\boxed{1}$

(1)

$|\vec{p}|^2$

$= |(1-t)\overrightarrow{OA} + t\overrightarrow{OB}|^2$

$= (1-t)^2 + t^2 = 2t^2 - 2t + 1$

$\therefore |\vec{p}| = \underline{\sqrt{2t^2-2t+1}}$ //

$|\vec{q}|$

$= \sqrt{(1-s)^2 2 + s^2 \cdot 2}$

$= \underline{\sqrt{4s^2-4s+2}}$ //

$\vec{p} \cdot \vec{q}$

$= t(1-s)(-1) + ts(-1)$

$= \underline{-t}$ //

(2)

$\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}| \cos \frac{3}{4}\pi$

$\Leftrightarrow -\frac{1}{2} = \sqrt{\frac{1}{2}} \sqrt{4s^2-4s+2} \left(-\frac{\sqrt{2}}{2}\right)$

$\downarrow$ 2乗

$1 = 4s^2 - 4s + 2$

$\Leftrightarrow s = \frac{1}{2}$

(3)

$|\vec{p} - \vec{q}|^2$

$= |\vec{p}|^2 - 2\vec{p} \cdot \vec{q} + |\vec{q}|^2$

$= 2t^2 - 2t + 1 + 2t + 4s^2 - 4s + 2$

$= 2t^2 + 4\left(s - \frac{1}{2}\right)^2 + 2$

$t = 0,\ s = \frac{1}{2}$ のとき

$\min |\vec{p} - \vec{q}| = \underline{\sqrt{2}}$ //

$\boxed{2}$

(1) $z + \frac{4}{z} \in \mathbb{R}$　$z \neq 0$

$\downarrow$

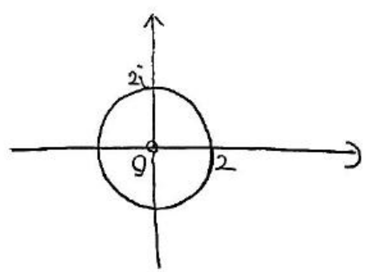
$z + \frac{4}{z} = \bar{z} + \frac{4}{\bar{z}}$

$\Leftrightarrow z^2\bar{z} + 4\bar{z} = z(\bar{z})^2 + 4z$

$\Leftrightarrow z\bar{z}(z - \bar{z}) - 4(z - \bar{z}) = 0$

$\Leftrightarrow (|z|^2 - 4)(z - \bar{z}) = 0$

$|z| = 2$ または $z = \bar{z}$　$(z \neq 0)$



(2) 与式の左辺が実数なので

$z - \frac{4}{z}$ が純虚数となればいい.

$z - \frac{4}{z} + \bar{z} - \frac{4}{\bar{z}} = 0$

$\Leftrightarrow z^2\bar{z} - 4\bar{z} + z\bar{z}^2 - 4z = 0$

$\Leftrightarrow |z|^2(z + \bar{z}) - 4(z + \bar{z}) = 0$

$\Leftrightarrow (|z|^2 - 4)(z + \bar{z}) = 0$

$|z| = 2$ または $z + \bar{z} = 0$.　$(z \neq 0)$

つまり条件にあうのは

$|z| = 2$

$\downarrow$

$z\bar{z} = 4 \Leftrightarrow \bar{z} = \frac{4}{z}$

より

$k(z + \bar{z} + 8) = i(z - \bar{z})$

$z = 2\cos\theta + 2(\sin\theta)i$ とすると

$k(4\cos\theta + 8) = i \cdot 4(\sin\theta)i$

$= -4\sin\theta$

$\Leftrightarrow k = \frac{-\sin\theta}{\cos\theta + 2} = f(\theta)$

$y = k$ と $y = f(\theta)$ が交点をもつような

k の範囲を答える.

$f'(\theta) = \frac{-\cos\theta(\cos\theta + 2) - (-\sin\theta)(-\sin\theta)}{(\cos\theta + 2)^2}$

$= \frac{-1 - 2\cos\theta}{(\cos\theta + 2)^2}$

θ	$0 \cdots \frac{2}{3}\pi \cdots \frac{4}{3}\pi \cdots 2\pi$
$f'(\theta)$	$-\ 0\ +\ 0\ -$
$f(\theta)$	$0 \searrow -\frac{\sqrt{3}}{3} \nearrow \frac{\sqrt{3}}{3} \searrow (0)$

求める範囲は

$\underline{-\frac{\sqrt{3}}{3} \leqq k \leqq \frac{\sqrt{3}}{3}}$ //

$\boxed{3}$

(1)

$1000a + 100b + 10c + d$ が 9 の倍数

ならば m を整数として

$1000a + 100b + 10c + d = 9m$

$\Leftrightarrow a + b + c + d = 9\{m - 111a - 11b - c\}$

となるので $a + b + c + d$ も 9 の倍数.

逆の場合も同様に言えるのでこれらは同値である.

(2)

$P(n$が9の倍数$)$

$=P($和が9の倍数$)$

$=P\begin{pmatrix}\text{組}\\(0,0,1,8)\\(0,2,8,8)\\(1,1,8,8)\end{pmatrix}$

$=\dfrac{1\cdot2\cdot2}{{}_8C_4}+\dfrac{2\cdot2\cdot1}{{}_8C_4}+\dfrac{1\cdot1}{{}_8C_4}$

$=\dfrac{9}{70}$ //

(3)

$P(n$が偶数$)$

$=1-P(1$が1番右にくる$)$

$=1-\dfrac{2\cdot{}_7P_3}{{}_8P_4}$ 1_aか1_bか

$=\dfrac{3}{4}$

$P_{n\text{が偶数}}(n$が9の倍数$)$

$=\dfrac{P(n\text{が}18\text{の倍数})}{P(n\text{が偶数})}$

$=\dfrac{4}{3}\left(\dfrac{2\cdot2\cdot3\cdot3!}{{}_8P_4}+\dfrac{4}{70}+\dfrac{2\cdot3!}{{}_8P_4}\right)$

$=\dfrac{4}{3}\left(\dfrac{72+96+12}{1680}\right)$

$=\dfrac{1}{7}$ //

$\boxed{4}$

(1) 与式を2乗

$BQ^2\geqq OQ^2\geqq 4AQ^2$

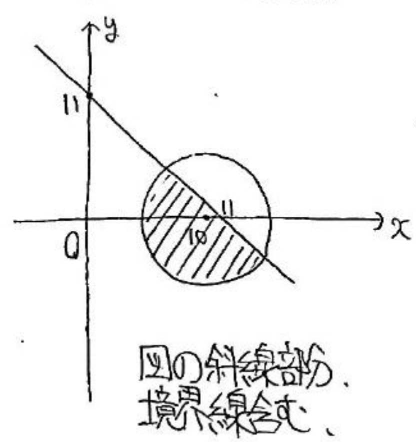
$\Leftrightarrow (x-11)^2+(y-11)^2$
$\geqq x^2+y^2\geqq 4\{(x-\frac{15}{2})^2+y^2\}$

$\Leftrightarrow$
$\underbrace{-22x-22y+242\geqq 0}_{①}\geqq \underbrace{3x^2-60x+3y^2+225}_{②}$

① $x+y-11\leqq 0$ $\Leftrightarrow y\leqq -x+11$

② $x^2-20x+y^2+\frac{225}{3}\leqq 0$ $\Leftrightarrow (x-10)^2+y^2\leqq 25$

①かつ②より Dの領域は



図の斜線部分.
境界線含む.

直線と円を連立

$x^2-20x+x^2-22x+121+75=0$

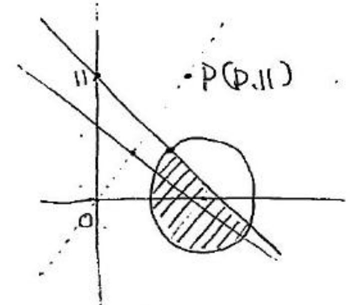
$\Leftrightarrow x^2-21x+98=0$

$\therefore x=7,14$

$BQ=OQ=2AQ$となるQは

$(7,4),(14,-3)$ //

(2)



OPの垂直二等分線は

$y=-\frac{p}{11}(x-\frac{p}{2})+\frac{11}{2}$

$=-\frac{p}{11}x+\frac{p^2}{22}+\frac{11}{2}$

図より $(7,4)$が $y\geqq -\frac{p}{11}x+\frac{p^2}{22}+\frac{11}{2}$

にあればいいので

$4\geqq -\frac{7}{11}p+\frac{p^2}{22}+\frac{11}{2}$

$\Leftrightarrow 88\geqq -14p+p^2+121$

$\Leftrightarrow 0\geqq p^2-14p+33$

$\Leftrightarrow 3\leqq p\leqq 11$ //

$\boxed{5}$

(1) $h(x)=\sin x-\frac{2}{\pi}x$ とおく.

$h'(x)=\cos x-\frac{2}{\pi}$

$h'(x)=0\Leftrightarrow x=\alpha\ (0\leqq\alpha\leqq\frac{\pi}{2})$

とおく.

x	$0\cdots\alpha\cdots\frac{\pi}{2}$
$h'(x)$	$+\ 0\ -$
$h(x)$	$0\nearrow\ \ \searrow 0$

増減表より $h(x)\geqq 0$

$\therefore \frac{2}{\pi}x\leqq\sin x$

(2)

$i(x)=g(x)-f(x)$ とおく

$i'(x)=\frac{1}{2}(\frac{\pi^2}{4}-x^2)^{-\frac{1}{2}}(-2x)+\sin x$

$=\sin x-\dfrac{x}{\sqrt{\frac{\pi^2}{4}-x^2}}$

$\geqq\left(\frac{2}{\pi}-\dfrac{1}{\sqrt{\frac{\pi^2}{4}-x^2}}\right)x$

$\geqq 0\quad (0\leqq x\leqq\frac{\pi}{2})$

これより$i(x)$は単調増加.

$i(\frac{\pi}{2})=0-(\frac{\pi}{2}-\frac{\pi}{2})=0$

$\therefore i(x)\leqq 0$

$\therefore g(x)\leqq f(x)$

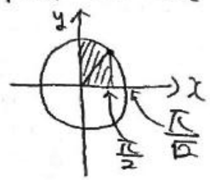
(3)

$$\int_0^{\frac{\pi}{2}} \left\{ \cos x - \left(\sqrt{\frac{\pi^2}{2} - x^2} - \frac{\pi}{2} \right) \right\} dx$$

$$= \left[\sin x + \frac{\pi}{2} x \right]_0^{\frac{\pi}{2}}$$

$$- \int_0^{\frac{\pi}{2}} \sqrt{\frac{\pi^2}{2} - x^2}\, dx$$

$$= 1 + \frac{\pi^2}{4} -$$



$$= 1 + \frac{\pi^2}{4} - \left(\frac{\pi}{\sqrt{2}} \cdot \frac{\pi}{\sqrt{2}} \cdot \pi \cdot \frac{1}{8} \right.$$

$$\left. + \frac{\pi}{2} \cdot \frac{\pi}{2} \cdot \frac{1}{2} \right)$$

$$= 1 + \frac{\pi^2}{8} - \frac{\pi^3}{16}$$ //