2017 横浜市立大

[I]

(1)

$$\begin{array}{r} 173 \\ 861 \overline{)148953} \\ 861 \\ \hline 6285 \\ 6027 \\ \hline 2583 \\ 2583 \\ \hline 0 \end{array} \qquad \begin{array}{r} 2 \\ 148953 \overline{)297767} \\ 297906 \\ \hline 861 \end{array}$$

$\dfrac{148953}{297767}$

$= \dfrac{861 \times 173}{2 \times 148953 + 861}$

$= \dfrac{173}{2 \times 173 + 1} = \dfrac{173}{347}$ //

(2) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{2\cos^2 x - 1} dx$　　$\cos x = t$, $-\sin x\, dx = dt$

$= \displaystyle\int_{\frac{1}{2}}^{0} \frac{1}{2t^2 - 1}(-dt)$

$= \displaystyle\int_0^{\frac{1}{2}} \left( \frac{1}{\sqrt{2}t - 1} - \frac{1}{\sqrt{2}t + 1} \right) \frac{1}{2} dt$

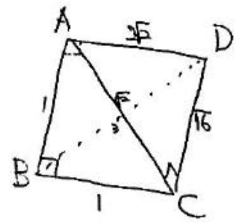
$= \dfrac{1}{2\sqrt{2}} \Big[ \log|\sqrt{2}t - 1| - \log|\sqrt{2}t + 1| \Big]_0^{\frac{1}{2}}$

$= \dfrac{1}{2\sqrt{2}} \log \dfrac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}$

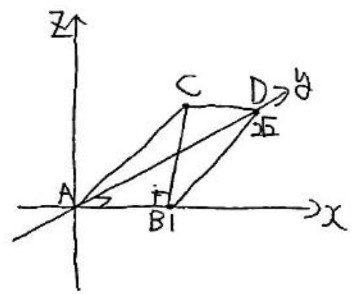
$= \dfrac{1}{2\sqrt{2}} \log \dfrac{\sqrt{2} - 1}{\sqrt{2} + 1}$

$= \dfrac{1}{\sqrt{2}} \log(\sqrt{2} - 1)$ //

(3)




△ABCが直角二等辺三角形,
△ACDが $1:\sqrt{5}:2$ の直角三角形だとわかる. 以下のようにA(0.0.0)、B(1.0.0), D(0, $2\sqrt{2}$, 0)とする.




∠ABC=90°より C(1, $y_c$, $z_c$)とおくと

$$\begin{cases} BC^2 = y_c^2 + z_c^2 = 1 \\ CD^2 = 1 + (y_c - 2\sqrt{2})^2 + z_c^2 = 6 \\ \Leftrightarrow y_c^2 - 4\sqrt{2}y_c + z_c^2 = -3 \end{cases}$$

∴ $y_c = \dfrac{1}{\sqrt{2}}$, $z_c = \dfrac{1}{\sqrt{2}}$

4点通る球の中心(x, y, z)とおく.
A.B.C.Dまでの距離の2乗が等しいので

$x^2 + y^2 + z^2 = (x-1)^2 + y^2 + z^2$
$= (x-1)^2 + \left(y - \frac{1}{\sqrt{2}}\right)^2 + \left(z - \frac{1}{\sqrt{2}}\right)^2 = x^2 + (y - 2\sqrt{2})^2 + z^2$

上の式より解くと

$x = \dfrac{1}{2}$, $y = \sqrt{2}$, $z = -\dfrac{1}{\sqrt{2}}$

∴ $R^2 = \dfrac{1}{4} + 2 + \dfrac{1}{2} = \dfrac{11}{4}$

∴ (半径) $= \dfrac{\sqrt{11}}{2}$ //

(4)

$f'(x) = -(1 + e^{-x})^{-2}(-e^{-x})$

$= \dfrac{e^{-x}}{(1 + e^{-x})^2}$

$= \dfrac{1}{e^{-x} + 2 + e^{x}}$

$\leqq \dfrac{1}{2\sqrt{e^{-x} \cdot e^{x}} + 2}$　（相加平均 ≧ 相乗平均）

$= \dfrac{1}{4}$

$x = 0$ のとき最大値 $\dfrac{1}{4}$ //

[II] (1)

$f(x)$

$= \sin\left(k + \frac{1}{2}\right)x - \sin\left(k - \frac{1}{2}\right)x$

$= \sin kx \cos\frac{x}{2} + \cos kx \sin\frac{x}{2}$
$\quad - \left( \sin kx \cos\frac{x}{2} - \cos kx \sin\frac{x}{2} \right)$

$= 2\cos kx \sin\frac{x}{2}$

$X = kx$, $Y = \dfrac{x}{2}$ //

(2) (与式)

$= \dfrac{\sin\left(m + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} - \displaystyle\sum_{k=1}^{m} \frac{\sin\left(k + \frac{1}{2}\right)x - \sin\left(k - \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$

$= \dfrac{\sin\left(m + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} - \dfrac{\sin\left(m + \frac{1}{2}\right)x - \sin\frac{x}{2}}{2\sin\frac{x}{2}}$

$= \dfrac{1}{2}$ //

(3)

(i) $n = 1$ のとき

$g_1(x) = \cos x \geqq -1$

より与式は成立.

(ii) $n=\alpha$ のとき与式が成り立つとする。

$n=\alpha+1$ のとき

$g_{\alpha+1}(x)$

$=\sum_{k=1}^{\alpha+1}(\alpha+2-k)\cos kx$

$=(\alpha+1)\cos x+\alpha\cos 2x+\cdots+\cos(\alpha+1)x$

$=\alpha\cos x+(\alpha-1)\cos 2x+\cdots+\cos\alpha x$

$+\cos x+\cos 2x+\cdots+\cos(\alpha+1)x$

$=g_\alpha(x)+\sum_{k=1}^{\alpha+1}\cos kx$

$=g_\alpha(x)+\dfrac{\sin(\alpha+\frac{3}{2})x}{2\sin\frac{1}{2}x}-\dfrac{1}{2}$

$=g_\alpha(x)-\dfrac{1}{2}+\dfrac{\sin(2\alpha+3)\frac{x}{2}}{2\sin\frac{x}{2}}$

ここで $\beta$ を整数として

$2\beta\pi<\frac{x}{2}<(2\beta+1)\pi$ のとき

$\sin\frac{x}{2}>0,\ \sin(2\alpha+3)\frac{x}{2}>0$

$(2\beta+1)\pi<\frac{x}{2}<(2\beta+2)\pi$ のとき

$\sin\frac{x}{2}<0,\ \sin(2\alpha+3)\frac{x}{2}<0$

より

$\dfrac{\sin(2\alpha+3)\frac{x}{2}}{2\sin\frac{x}{2}}>0$

よって

$g_{\alpha+1}(x)$

$>g_\alpha(x)-\frac{1}{2}$

$\geqq-\frac{\alpha+1}{2}-\frac{1}{2}=-\frac{\alpha+1+1}{2}$

よりこのときも成立。

(i)(ii)より $g_n(x)\geqq-\dfrac{n+1}{2}$

[III]

(1)

ちょうど $2n$ 回投げてAが勝つ確率は、

2回で勝負が決まらない確率は

$1-2p(1-p)$

より

$\{1-2p(1-p)\}^{n-1}p(1-p)$

一方でちょうど $2n$ 回投げてBが勝つ確率は

$\{1-2p(1-p)\}^{n-1}(1-p)p$

より両者は等しい。

(2)

求める確率は

$2p(1-p)+\{1-2p(1-p)\}2p(1-p)$

$+\cdots+\{1-2p(1-p)\}^{n-1}2p(1-p)$

$=\dfrac{1-\{1-2p(1-p)\}^n}{1-\{1-2p(1-p)\}}2p(1-p)$

$=1-\{1-2p(1-p)\}^n$

$=1-(2p^2-2p+1)^n$

(3)

$1-(2p^2-2p+1)^n$

$=1-\{2(p-\frac{1}{2})^2+\frac{1}{2}\}^n$

$\leqq1-(\frac{1}{2})^n$

等号成立は $p=\frac{1}{2}$、このとき最大。

(4)

$0<nx^n$

$=\dfrac{n}{(\frac{1}{x})^n}$

$=\dfrac{n}{(1+\frac{1}{x}-1)^n}$

$=\dfrac{n}{\sum_{k=0}^{n}{}_nC_k\cdot1^{n-k}(\frac{1}{x}-1)^k}$

$<\dfrac{n}{{}_nC_0 1^n+{}_nC_1 1^{n-1}(\frac{1}{x}-1)+{}_nC_2 1^{n-2}(\frac{1}{x}-1)^2}$

$=\dfrac{1}{\frac{1}{n}+(\frac{1}{x}-1)+\frac{n-1}{2}(\frac{1}{x}-1)^2}$

$=f(n)$ とおく

$\lim_{n\to\infty}f(n)=0$ より

$\lim_{n\to\infty}nx^n=0$.

(5) $S_n=\sum_{i=1}^{n}2ia_i$ とおく。

$S_n=2a_1+4a_2+\cdots+2na_n$

$-\{1-2p(1-p)\}S_n=\quad 2a_2+\cdots+2(n-1)a_n+2na_{n+1}$

$2p(1-p)S_n=2a_1+2a_2+\cdots+2a_n-2na_{n+1}$

$=2\times(3)-2na_{n+1}$

$\therefore S_n=\dfrac{1-(2p^2-2p+1)^n-n\{1-2p(1-p)\}^n2p(1-p)}{p(1-p)}$

$\lim_{n\to\infty}S_n=\dfrac{1}{p(1-p)}$ ($\because$(4))

[IV]

(1)

(左辺)

$= P'(t) = -r\left(P_0 - \frac{K}{r}\right)e^{-rt}$

(右辺)

$= K - r\left\{\frac{K}{r} + \left(P_0 - \frac{K}{r}\right)e^{-rt}\right\}$

$= -r\left(P_0 - \frac{K}{r}\right)e^{-rt}$

(2)

$\begin{cases} P(3) = \alpha + (5-\alpha)e^{-3r} = 11 \\ P(9) = \alpha + (5-\alpha)e^{-9r} = 17 \end{cases}$

$\Downarrow$

$\begin{cases} e^{-3r} = \frac{11-\alpha}{5-\alpha} \\ e^{-9r} = \frac{17-\alpha}{5-\alpha} \end{cases}$

$\Downarrow$

$\left(\frac{11-\alpha}{5-\alpha}\right)^3 = \frac{17-\alpha}{5-\alpha}$

$\Leftrightarrow (11-\alpha)^3 = (17-\alpha)(5-\alpha)^2$

$\Leftrightarrow 11^3 - 3\cdot121\alpha + 3\cdot11\alpha^2 - \alpha^3$

$= 17\cdot25 - 170\alpha + 17\alpha^2$

$-25\alpha + 10\alpha^2 - \alpha^3$

$\Leftrightarrow 6\alpha^2 - 168\alpha + 11^3 - 17\cdot25 = 0$

$\Leftrightarrow 6\alpha^2 - 168\alpha + 906 = 0$

$\Leftrightarrow \alpha^2 - 28\alpha + 151 = 0$

$\therefore \underline{\alpha = 14 + 3\sqrt{5}}$ //

($\because \alpha = 14 - 3\sqrt{5}$ だと $e^{-3r}, e^{-9r} > 0$ とならない)

(3)

$e^{-3r} = \frac{-3-3\sqrt{5}}{-9-3\sqrt{5}}$

$= \frac{1+\sqrt{5}}{3+\sqrt{5}}$

$= \frac{\sqrt{5}-1}{2}$

$\Leftrightarrow -3r = \log \frac{\sqrt{5}-1}{2}$

$\therefore \underline{r = -\frac{1}{3}\log \frac{\sqrt{5}-1}{2}}$ //

(4)

$P(6)$

$= \alpha + (5-\alpha)e^{-6r}$

$= 14 + 3\sqrt{5} + (-9-3\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^2$

$= \underline{8 + 3\sqrt{5}}$ //