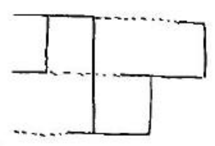
2017 杏林医

I



(a)

3回で優勝が決まらない場合
4回目はAとBが対戦 ⋯ ①

P(6回でCが優勝)

$= P(A \to C \to B \to A \to C \to C)$
$+ P(B \to C \to A \to B \to C \to C)$

$= \frac{1}{2}(1-P)P\frac{1}{2}(1-P)^2$
$+ \frac{1}{2}(1-P)P\frac{1}{2}(1-P)^2$

$= \frac{1}{2}(1-P)^3 P = f(P)$ とおく

$f'(P) = \frac{1}{2}(-3)(1-P)^2 P + \frac{1}{2}(1-P)^3$
$= \frac{1}{2}(1-P)^2(1-4P)$

$P = \frac{1}{4}$ の前後で $f'(P)$ が $+ \to -$
なので $P = \frac{1}{4}$ のとき最大値 $\frac{27}{512}$

AとBは対等なので
$\alpha = \beta$ ⋯ ②

(b)

P(3回でCが優勝)
$= P(A \to C \to C) + P(B \to C \to C)$
$= \frac{1}{2}(1-P)^2 + \frac{1}{2}(1-P)^2$
$= (1-P)^2$ $\quad P = \frac{1}{3}$
$= \frac{4}{9}$

P(Cが優勝)
$= P(3回で優勝) + P(6回で 〃)$
$+ P(9回で 〃) + \cdots$
$= (1-P)^2 + \frac{P}{2}(1-P)\cdot(1-P)^2 + \cdots$
$= \frac{(1-P)^2}{1-\frac{P}{2}(1-P)}$ ⋯ △1 $\quad P = \frac{1}{3}$
$= \frac{\frac{4}{9}}{1-\frac{1}{9}}$
$= \frac{1}{2}$

(c)

P(Aが優勝)
$= P(A \to A) + P(B \to C \to A \to A)$
$+ P(A \to C \to B \to A \to A)$
$+ P(B \to C \to A \to B \to C \to A \to A)$
$+ \cdots$

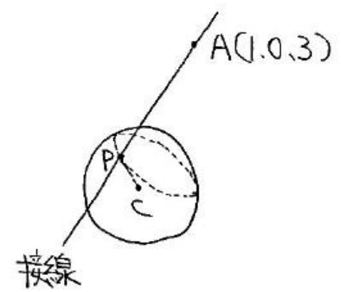
$= \frac{\frac{1}{2}P}{1-\frac{P}{2}(1-P)} + \frac{\frac{1}{2}(1-P)P\frac{1}{2}}{1-\frac{P}{2}(1-P)}$ ⋯ △2

△1 = △2 より
$(1-P)^2 = \frac{P}{2} + \frac{P}{4}(1-P)$
$\Leftrightarrow 5P^2 - 11P + 4 = 0$
$\therefore P = \frac{11-\sqrt{41}}{10}$

II

(a)

Pは球面S上にあるので
$x^2 + y^2 + (z-1)^2 = 1$ ⋯ ①

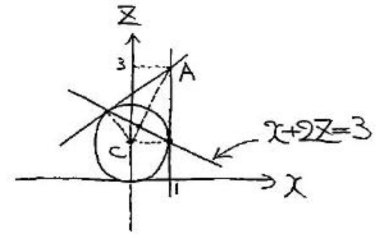


$\overrightarrow{CP} = \begin{pmatrix} x \\ y \\ z-1 \end{pmatrix}$ $\quad \overrightarrow{AP} = \begin{pmatrix} x-1 \\ y \\ z-3 \end{pmatrix}$

$\overrightarrow{CP} \cdot \overrightarrow{AP} = 0$ より
$x(x-1) + y^2 + (z-1)(z-3) = 0$
$\Leftrightarrow x^2 - x + y^2 + z^2 - 2z + 3 = 0$ ⋯ ②

①②辺々引いて整理すると
$x + 2z = 3$

y成分は関与してこないので
xz平面で考える．



平面と球面Sの交線は円になる
が中心は直線CA: $z = 2x + 1$
と $x + 2z = 3$ を連立すると
$x = \frac{1}{5}$, $z = \frac{7}{5}$

つまり中心 $\left(\frac{1}{5}, 0, \frac{7}{5}\right)$
半径rは
$r = \sqrt{\left(1-\frac{1}{5}\right)^2 + \left(1-\frac{7}{5}\right)^2}$
$= \sqrt{\frac{20}{25}} = \frac{2\sqrt{5}}{5}$

左図で考えれば三平方の定理より
$|\overrightarrow{AP}|^2 = CA^2 - 1^2$
$= 5 - 1 = 4$
$\therefore |\overrightarrow{AP}| = 2$

(b)

$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ}$

$\Leftrightarrow \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + k \begin{pmatrix} x-1 \\ y \\ z-3 \end{pmatrix}$

$= \begin{pmatrix} 1+k(2-2z) \\ ky \\ 3+k(z-3) \end{pmatrix}$

より

$k(z-3) = -3$

$z-3 = -\dfrac{3}{k} \quad (k \neq 0)$

$z = 3 - \dfrac{3}{k}$

$X = 1 + k\left\{2 - 2\left(3 - \dfrac{3}{k}\right)\right\}$

$= 1 + k\left(\dfrac{6}{k} - 4\right)$

$\Leftrightarrow 4k = 7 - X$

$\therefore k = \underline{\dfrac{7-X}{4}}$

$|\overrightarrow{AQ}|^2 = k^2 |\overrightarrow{AP}|^2$

$\Leftrightarrow (X-1)^2 + Y^2 + 9 = 4k^2$

$\Leftrightarrow X^2 - 2X + Y^2 + 10 = \dfrac{(7-X)^2}{4}$

整理して

$\underline{X^2 + 2X + \dfrac{4}{3}Y^2 = 3}$

$\Leftrightarrow (X+1)^2 + \dfrac{4}{3}Y^2 = 4$

$\Leftrightarrow \dfrac{(X+1)^2}{4} + \dfrac{Y^2}{3} = 1$

焦点の X+1座標, Y座標は

$(\pm\sqrt{4-3}, 0)$

焦点の X座標, Y座標は

$(-1 \pm 1, 0)$

よって原点と $(-2, 0)$

$\hookrightarrow \underline{(-2, 0, 0)}$

Ⅲ

(1) $\lim_{\theta \to \frac{\pi}{3}}$ (分母) $= 0$ より

$\lim_{\theta \to \frac{\pi}{3}}$ (分子) $= -1 - \dfrac{k}{2} = 0$

$\therefore k = \underline{-2}$

このとき

(与式)

$= \lim_{\theta \to \frac{\pi}{3}} \dfrac{4\cos^3\theta - 3\cos\theta - 2(2\cos^2\theta - 1)}{2 + 2\cos\theta - 3}$

$= \lim_{\theta \to \frac{\pi}{3}} \dfrac{4\cos^3\theta - 4\cos^2\theta - 3\cos\theta + 2}{2\cos\theta - 1}$

$= \lim_{\theta \to \frac{\pi}{3}} (2\cos^2\theta - \cos\theta - 2)$

$= \dfrac{1}{2} - \dfrac{1}{2} - 2 = \underline{-2}$

(2)

$f(x)$

$= \lim_{\alpha \to 2} \dfrac{\overbrace{\boxed{\cos 2x\alpha}}^{g(\alpha)} - \cos x \cdot 2}{\alpha - 2} \cdot \dfrac{1}{\alpha + 3}$

$= g'(2)\dfrac{1}{5}$ $\quad \boxed{g'(\alpha) = -x\sin x\alpha}$

$= -\dfrac{x}{5}\sin 2x$

$f\left(\dfrac{\pi}{6}\right)$

$= \underline{\dfrac{-\sqrt{3}}{60}\pi}$

$\lim_{x \to 0} \dfrac{f(x)}{x^2}$

$= \lim_{x \to 0} \left(-\dfrac{\sin 2x}{5x}\right)$

$= \lim_{x \to 0} \left(-\dfrac{\sin 2x}{2x} \cdot \dfrac{2}{5}\right) = \underline{\dfrac{-2}{5}}$

$\int_0^{\frac{\pi}{4}} f(x)\,dx$

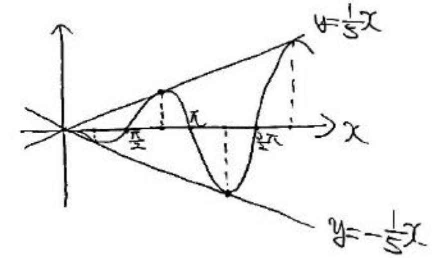
$= -\dfrac{1}{5}\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$

$= -\dfrac{1}{5}\left[-\dfrac{x}{2}\cos 2x + \dfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \underline{\dfrac{-1}{20}}$

$f(x) = -\dfrac{x}{5} \times \sin 2x$ より

$-1 \leqq \sin 2x \leqq 1$ であることを考えると



$f(x)$は $y = \pm\dfrac{1}{5}x$ を基準に

図を書くことになる.

これより $y = ax + b$ と共有点を持た

ないのは

$a > \dfrac{1}{5}$ かつ $b > 0$

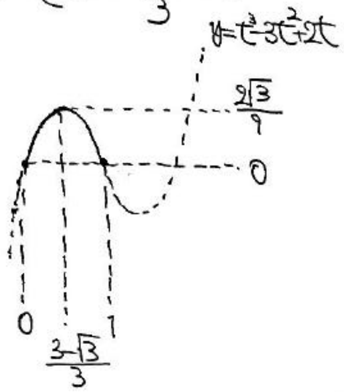
または $a < \dfrac{-1}{5}$ かつ $b < 0$

IV

(a)

$\frac{dy}{dt} = 3t^2 - 6t + 2 = 0$

$\Leftrightarrow t = \frac{3 \pm \sqrt{3}}{3}$



$\underline{t = \frac{3-\sqrt{3}}{3} \text{ のとき最大値 } \frac{2\sqrt{3}}{9}}$ //

$\frac{dy}{dx} = \frac{3t^2 - 6t + 2}{-2\sqrt{3}t + 2\sqrt{3}} = \tan\left(\pm\frac{\pi}{6}\right) = \pm\frac{1}{\sqrt{3}}$

解くと

$\underline{t = \frac{2}{3}}$ // $(\because 0 < t < 1)$

のとき x軸とのなす鋭角が $\frac{\pi}{6}$.

(b)

$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(-2\sqrt{3}t + 2\sqrt{3})^2 + (3t^2 - 6t + 2)^2}$

$= \sqrt{9t^4 - 36t^3 + 60t^2 - 48t + 16}$

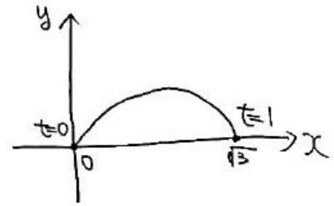
$= \underline{3t^2 - 6t + 4}$ // $\boxed{\text{暗算でがんばる}}$

(Cの長さ)

$= \int_0^1 v\,dt$

$= \left[t^3 - 3t^2 + 4t\right]_0^1$

$= \underline{2}$ //



$\int_0^{\sqrt{3}} y\,dx$

$= \int_0^1 y \frac{dx}{dt}\,dt$

$= \int_0^1 (t^3 - 3t^2 + 2t)(-2\sqrt{3}t + 2\sqrt{3})\,dt$

$= -2\sqrt{3}\int_0^1 t(t-1)(t-2)(t-1)\,dt$

$= -2\sqrt{3}\int_0^1 (t^2 - 2t + 1 - 1)(t-1)^2\,dt$

$= -2\sqrt{3}\int_0^1 \left\{(t-1)^4 - (t-1)^2\right\}dt$

$= -2\sqrt{3}\left[\frac{1}{5}(t-1)^5 - \frac{1}{3}(t-1)^3\right]_0^1$

$= -2\sqrt{3}\left(\frac{1}{5} - \frac{1}{3}\right)$

$= \underline{\frac{4\sqrt{3}}{15}}$ //