2017 国際医療福祉大

第1問

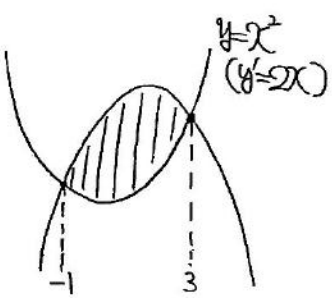
(A)

$x = 10100_{(3)}$
$= 1\cdot 3^4 + 1\cdot 3^2 = \underline{90_{(10)}}$ //

xy が3進法で8ケタ
$\Leftrightarrow 3^7 \leqq xy < 3^8$
$\Leftrightarrow \frac{1}{10}\cdot \underbrace{3^5}_{243} \leqq y < \frac{1}{10}\underbrace{3^6}_{729}$

$\therefore \underline{25 \leqq y \leqq 72}$ //

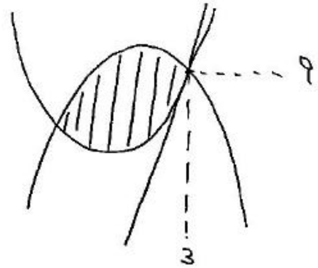
(B)



$ax + y = k$
$\Leftrightarrow y = -ax + k$ とおく.

$y=x^2$ の $x=-1, 3$ での微分係数は $-2, 6$ なのでこれと $-a$ を比較する.

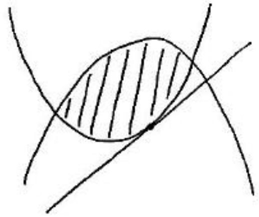
(i) $-a \geqq 6 \Leftrightarrow a \leqq \underline{-6}$ のとき //



$(3, 9)$ のとき

$\min k = m = \underline{3a+9}$ //

(ii) $-2 < -a < 6 \Leftrightarrow -6 < a < \underline{2}$ のとき //

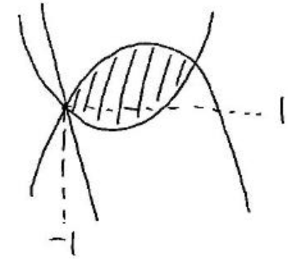


$y = x^2$ と $y = -ax + k$ を連立.
$x^2 = -ax + k$
$\Leftrightarrow x^2 + ax - k = 0$
$D = a^2 - 4(-k) = 0$
$\therefore k = -\frac{a^2}{4}$

$\left(-\frac{a}{2}, \frac{a^2}{4}\right)$ のとき

$\min k = m = \underline{\frac{-1}{4}a^2}$ //

(iii) $-a \leqq -2 \Leftrightarrow 2 \leqq a$ のとき



$(-1, 1)$ のとき

$\min k = m = \underline{-a+1}$ //

(C)

$\alpha = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= 2\left(\cos\underline{\frac{\pi}{3}} + i\sin\frac{\pi}{3}\right)$ //

$\alpha^3 = 2^3(\cos\pi + i\sin\pi)$
$= \underline{-8}$ //

z_n
$= \frac{1}{2i}\left\{2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) - 2^n\left(\cos\frac{(-n\pi)}{3} + i\sin\frac{(-n\pi)}{3}\right)\right\}$
$= \frac{1}{2i}\left(2\cdot 2^n i\sin\frac{n\pi}{3}\right)$
$= 2^n \sin\frac{n\pi}{3}$

$\sum_{k=1}^{30} z_k$
$= z_1 + z_7 + z_{13} + z_{19} + z_{25}$
$+ (z_2 + z_8 + z_{14} + z_{20} + z_{26})$
$+ (z_4 + z_{10} + z_{16} + z_{22} + z_{28})$
$+ (z_5 + z_{11} + z_{17} + z_{23} + z_{29})$
$=$ (初項 $z_1+z_2+z_4+z_5$ 公比 $2^6=64$, 項数5の等比数列の和)

ここで
$z_1 + z_2 + z_4 + z_5$
$= \sqrt{3} + 2\sqrt{3} - 8\sqrt{3} - 16\sqrt{3} = -21\sqrt{3}$

$= \frac{-21\sqrt{3} - (-21\sqrt{3}\cdot 64^5)}{1-64}$
$= \frac{-21\sqrt{3}}{-63}(1-64^5)$
$= \underline{\frac{\sqrt{3}}{3}(1-2^{30})}$ //

第2問

(1)

200回のうち奇数回目には A，偶数回目には A以外が赤を持っているべき．

赤の変遷は

$$A \xrightarrow{\frac{1}{3}} B \xrightarrow{\frac{1}{3}} A$$
$$A \xrightarrow{\frac{2}{3}} C \xrightarrow{\frac{2}{3}} A$$

以上のような確率になる．

つまり

$$P_k = {}_{100}C_k\left(\frac{1}{9}\right)^k\left(\frac{4}{9}\right)^{100-k}$$

また

$$\frac{P_{k+1}}{P_k} = \frac{\frac{100!}{(k+1)!(99-k)!}\left(\frac{1}{9}\right)^{k+1}\left(\frac{4}{9}\right)^{99-k}}{\frac{100!}{k!(100-k)!}\left(\frac{1}{9}\right)^{k}\left(\frac{4}{9}\right)^{100-k}}$$
$$= \frac{100-k}{k+1}\cdot\frac{1}{9}\cdot\frac{9}{4}$$
$$= \frac{100-k}{4k+4} > 1$$
$$\Leftrightarrow 100-k > 4k+4$$
$$\Leftrightarrow k < \frac{96}{5}$$

つまり

$$\frac{P_{k+1}}{P_k} > 1 \Leftrightarrow k < \frac{96}{5} \quad \therefore k \leqq 19$$
$$\frac{P_{k+1}}{P_k} < 1 \Leftrightarrow k > \frac{96}{5} \quad \therefore k \geqq 20$$

よって

$$P_0 < P_1 < \cdots < P_{19} < P_{20} > P_{21} > \cdots$$

最大となるkは $k=20$

(2)

n回目 → n+2回目

A　$a_n \xrightarrow{\frac{1}{3}\cdot\frac{1}{3}+\frac{2}{3}\cdot\frac{2}{3}} a_{n+2}$

A以外　$1-a_n$ → $\frac{1}{3}\cdot\frac{2}{3}$ (または $\frac{2}{3}\cdot\frac{1}{3}$)

以上より

$$a_{n+2} = \frac{5}{9}a_n + \frac{2}{9}(1-a_n)$$
$$= \frac{1}{3}a_n + \frac{2}{9}$$

特性方程式

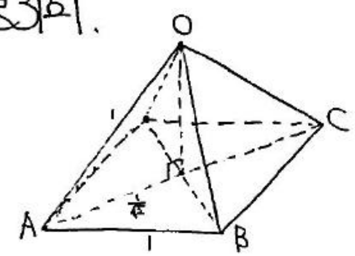
$$\alpha = \frac{1}{3}\alpha + \frac{2}{9}$$
$$\therefore \alpha = \frac{1}{3}$$

$$a_{n+2} - \frac{1}{3} = \frac{1}{3}\left(a_n - \frac{1}{3}\right)$$

$$\therefore a_{2m} - \frac{1}{3} = (\underbrace{a_2}_{\frac{5}{9}} - \frac{1}{3})\left(\frac{1}{3}\right)^{m-1}$$

$$\therefore a_{2m} = \frac{2}{9}\left(\frac{1}{3}\right)^{m-1} + \frac{1}{3}$$
$$= \frac{1}{3} + 2\left(\frac{1}{3}\right)^{m+1}$$

第3問.



(1) 高さは $\frac{1}{\sqrt{2}}$ より

$$1 \times \frac{1}{\sqrt{2}} \times \frac{1}{3} = \frac{\sqrt{2}}{6}$$

(2)

$$\frac{\overrightarrow{OD}+\overrightarrow{OB}}{2} = \frac{\overrightarrow{OA}+\overrightarrow{OC}}{2}$$

より

$$\overrightarrow{OD} = \overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC}$$
$$\therefore \overrightarrow{OS} = \frac{1}{2}\overrightarrow{OA} - \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}$$

(3)

$$\overrightarrow{OS} = \alpha\overrightarrow{OP} + \beta\overrightarrow{OR} + (1-\alpha-\beta)\overrightarrow{OQ}$$
$$= \underbrace{\alpha p}_{\frac{1}{2}}\overrightarrow{OA} + \underbrace{\beta r}_{\frac{1}{2}}\overrightarrow{OC} + \underbrace{\frac{2}{3}(1-\alpha-\beta)}_{-\frac{1}{2}}\overrightarrow{OB}$$

$$\alpha = \frac{1}{2p},\ \beta = \frac{1}{2r}$$

より

$$\frac{2}{3}\left(1 - \frac{1}{2p} - \frac{1}{2r}\right) = -\frac{1}{2}$$
$$\Leftrightarrow 1 - \frac{1}{2p} - \frac{1}{2r} = -\frac{3}{4}$$
$$\Leftrightarrow -\frac{1}{2}\left(\frac{1}{p}+\frac{1}{r}\right) = -\frac{7}{4}$$
$$\therefore \frac{1}{p}+\frac{1}{r} = \frac{7}{2}$$

(4)

$$r - p = pr$$
$$\Leftrightarrow \frac{1}{p} - \frac{1}{r} = 1$$
$$\therefore \frac{2}{p} = \frac{9}{2} \quad \frac{2}{r} = \frac{5}{2}$$
$$\therefore p = \frac{4}{9} \quad r = \frac{4}{5}$$

$$\overrightarrow{QP}\cdot\overrightarrow{QR}$$
$$= (\overrightarrow{OP}-\overrightarrow{OQ})\cdot(\overrightarrow{OR}-\overrightarrow{OQ})$$
$$= \left(\frac{4}{9}\vec{a} - \frac{2}{3}\vec{b}\right)\cdot\left(\frac{4}{5}\vec{c} - \frac{2}{3}\vec{b}\right)$$
$$= -\frac{8}{27}\underbrace{\vec{a}\cdot\vec{b}}_{\frac{1}{2}} - \frac{8}{15}\underbrace{\vec{b}\cdot\vec{c}}_{\frac{1}{2}} + \frac{4}{9}$$
$$= \frac{4}{135}$$

三角錐 OPQR

$= \frac{4}{9}\cdot\frac{2}{3}\cdot\frac{4}{5}\times\frac{1}{2}\times(1) = \frac{16}{135}(1)$

三角錐 OPRS

$= \frac{4}{9}\cdot\frac{4}{5}\cdot\frac{1}{2}\times\frac{1}{2}\times(1) = \frac{4}{45}(1)$

以上より

四角錐 O-PQRS

$= \left(\frac{16}{135}+\frac{4}{45}\right)\cdot\frac{\sqrt{2}}{6} = \underline{\frac{14\sqrt{2}}{405}}$ //

第4問

(1) $f_a(0)$

$= \int_0^a 0\,dt$

$= \underline{0}$ //

(2)

$f_a(x) = \int_0^a \left(\underline{\frac{-1}{2}}e^{-2t}\right)' \sin xt\,dt$

$f_a(x) = \int_0^a e^{-2t}\left(\underline{\frac{-1}{x}}\cos xt\right)' dt$

$f_a(x)$ に関して $f_a(x)$

$= \left[\frac{-1}{2}e^{-2t}\sin xt\right]_0^a - \int_0^a \frac{-x}{2}e^{-2t}\cos xt\,dt$

$f_a(x) = \left[e^{-2t}\frac{-1}{x}\cos xt\right]_0^a - \int_0^a \frac{2}{x}e^{-2t}\cos xt\,dt$

$\Leftrightarrow$

$$\begin{cases} f_a(x) = -\frac{1}{2}e^{-2a}\sin xa + \frac{x}{2}\int_0^a e^{-2t}\cos xt\,dt \\ f_a(x) = -\frac{1}{x}e^{-2a}\cos xa + \frac{1}{x} - \frac{2}{x}\int_0^a e^{-2t}\cos xt\,dt \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \int_0^a e^{-2t}\cos xt\,dt = \frac{2}{x}\left(f_a(x) + \frac{1}{2}e^{-2a}\sin xa\right) \\ \int_0^a e^{-2t}\cos xt\,dt = -\frac{x}{2}\left(f_a(x) + \frac{1}{x}e^{-2a}\cos xa - \frac{1}{x}\right) \end{cases}$$

$\therefore$

$\left(\frac{2}{x}+\frac{x}{2}\right)f_a(x)$

$= \underline{\frac{1}{2} - e^{-2a}\left(\frac{\sin ax}{x} + \frac{\cos ax}{2}\right)}$ //

(3)

(i) $x \neq 0$ のとき

$f(x) = \frac{\frac{1}{2}}{\frac{2}{x}+\frac{x}{2}} = \frac{x}{x^2+4}$

$= \frac{1}{2\left(\frac{2}{x}+\frac{x}{2}\right)}$

$\leq \frac{1}{2\cdot 2\sqrt{\frac{2}{x}\cdot\frac{x}{2}}}$ $\boxed{(相加平均) \geq (相乗平均)}$

$= \frac{1}{4}$

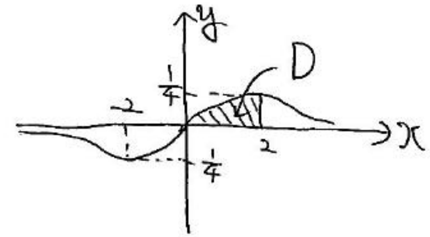
等号成立は

$\frac{2}{x} = \frac{x}{2} \Leftrightarrow x = \underline{2}$ // のとき

このとき 極大値 $\underline{\frac{1}{4}}$ //

(ii) $x = 0$ のとき (1)より

$f(x) = 0$



(Dの面積)

$= \int_0^2 \frac{x}{x^2+4}dx$

$= \left[\frac{1}{2}\log(x^2+4)\right]_0^2$

$= \frac{1}{2}\log 8 - \frac{1}{2}\log 4 = \underline{\log\sqrt{2}}$ //

(回転体)

$= \int_0^2 \left(\frac{x}{x^2+4}\right)^2 \pi\,dx$

$= \pi\int_0^2 \frac{x^2}{(x^2+4)^2}dx$

$x = 2\tan\theta$ とおく

$dx = \frac{2}{\cos^2\theta}d\theta$

$= \pi\int_0^{\frac{\pi}{4}} \frac{4\tan^2\theta}{\frac{16}{\cos^4\theta}}\cdot\frac{2}{\cos^2\theta}d\theta$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta$

$= \frac{\pi}{2}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left(\frac{\pi}{8} - \frac{1}{4}\right)$

$= \underline{\frac{\pi^2}{16} - \frac{\pi}{8}}$ //