2017 慶應理工

1.

$y=\sin(\omega t+\frac{2}{3}\pi)$
$=\cos(\omega t+\frac{\pi}{6})$

$\boxed{\sin(\theta+\frac{\pi}{2})=\cos\theta}$

$\downarrow$

$x=2y$

$\therefore \underline{y=\frac{1}{2}x}_{\#}\ (-2\leqq x\leqq 2)$

$x=2\cos(\omega t+\alpha+\frac{\pi}{6}-\alpha)$
$=2\cos\{\omega t+\alpha-(\alpha-\frac{\pi}{6})\}$
$=2\cos(\omega t+\alpha)\cos(\alpha-\frac{\pi}{6})$
$\quad+2\sin(\omega t+\alpha)\sin(\alpha-\frac{\pi}{6})$
$=2\cos(\omega t+\alpha)\sqrt{1-\beta^2}+2y\beta$

$\Leftrightarrow$
$x-2y\beta=2\cos(\omega t+\alpha)\sqrt{1-\beta^2}$

2乗すると

$x^2-4\beta xy+4\beta^2y^2$
$=4(1-y^2)(1-\beta^2)$
$=4(1-\beta^2-y^2+\beta^2y^2)$

$\therefore x^2-4\beta xy+4y^2=4(1-\beta^2)$

$\therefore \underline{\frac{x^2}{4(1-\beta^2)}-\frac{\beta}{1-\beta^2}xy+\frac{1}{1-\beta^2}y^2=1}_{\#}$

(2)

$\underline{|z|^2=a^2+b^2}_{\#}$

$\frac{5}{z}=\frac{5}{a+bi}$
$=\frac{5a}{a^2+b^2}-\frac{5b}{a^2+b^2}i$

$\frac{5a}{a^2+b^2}=k\in\mathbb{Z},\ \frac{5b}{a^2+b^2}=l\in\mathbb{Z}$

$\therefore \begin{cases}5a=k(a^2+b^2)\\5b=l(a^2+b^2)\end{cases}$

$|z|^2=a^2+b^2\neq 1$

$|z|\neq 5\Leftrightarrow |z|^2=a^2+b^2\neq 25$

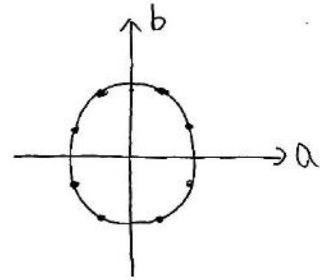
kやlが5の倍数だと

$k=5s,\ l=5t\ (s,t\in\mathbb{Z})$

として

$a=s(a^2+b^2)$
$b=t(a^2+b^2)$

これをみたす整数a,bは0しか存在しないのでNG. kとlが整数ならば $\underline{|z|^2=a^2+b^2=5}_{\#}$



$(a,b)=(\pm1,\pm2),(\pm2,\pm1)$

(複合任意)で $\underline{8個}_{\#}$

$(イ)=\underline{2+i}_{\#}$

(3) $y=\int_0^x e^{t^2}dt$

$\updownarrow$ 逆関数

$x=\int_0^y e^{t^2}dt$

yで微分

$\frac{dx}{dy}=e^{y^2}$

$\Leftrightarrow \frac{1}{\frac{dy}{dx}}=e^{y^2}$

$\therefore \frac{1}{g(x)}=e^{y^2}$

xで微分

$-\frac{g'(x)}{(g(x))^2}=2ye^{y^2}\frac{dy}{dx}=2y$

$\therefore g'(x)=-2y(g(x))^2$
$=-2ye^{-2y^2}$

$\therefore \underline{G(x)=-2xe^{-2x^2}}_{\#}$

2.

(1)

$\triangle ABC$
$=\frac{1}{2}\sqrt{\underbrace{|\overrightarrow{AB}|^2}_{10}\underbrace{|\overrightarrow{AC}|^2}_{4}-\underbrace{(\overrightarrow{AB}\cdot\overrightarrow{AC})^2}_{4}}$
$=\underline{3}_{\#}$

$|\overrightarrow{AB}|^2=10$
$\Leftrightarrow |\overrightarrow{OB}|^2-2\overrightarrow{OA}\cdot\overrightarrow{OB}+|\overrightarrow{OA}|^2=10$
$\Leftrightarrow r^2-\overrightarrow{OA}\cdot\overrightarrow{OB}=5$

同様に $|\overrightarrow{AC}|^2=4$ より
$r^2-\overrightarrow{OC}\cdot\overrightarrow{OA}=2$

$\overrightarrow{AB}\cdot\overrightarrow{AC}$
$=(\overrightarrow{OB}-\overrightarrow{OA})\cdot(\overrightarrow{OC}-\overrightarrow{OA})$
$=\overrightarrow{OB}\cdot\overrightarrow{OC}-(r^2-5)-(r^2-2)+r^2$
$=\overrightarrow{OB}\cdot\overrightarrow{OC}-r^2+7=-2$

$\therefore \overrightarrow{OB}\cdot\overrightarrow{OC}=\underline{r^2-9}_{\#}$

(2)

$\overrightarrow{OP}$ は $\overrightarrow{AB}, \overrightarrow{AC}$ と垂直

$(-3\overrightarrow{OA}+x\overrightarrow{OB}+y\overrightarrow{OC})(\overrightarrow{OB}-\overrightarrow{OA})=0$

$(-3\overrightarrow{OA}+x\overrightarrow{OB}+y\overrightarrow{OC})(\overrightarrow{OC}-\overrightarrow{OA})=0$

$\downarrow$

$-3(r^2-5)+xr^2+y(r^2-9)$
$+3r^2-x(r^2-5)-y(r^2-2)=0$

$-3(r^2-2)+x(r^2-9)+yr^2$
$+3r^2-x(r^2-5)-y(r^2-2)=0$

$\downarrow$

$15+5x-7y=0$

$6-4x+2y=0$

解くと $\underline{x=4,\ y=5}$ //

$\overrightarrow{AH}$

$=\overrightarrow{OH}-\overrightarrow{OA}$

$=\overrightarrow{OD}+\overrightarrow{DH}-\overrightarrow{OA}$

$=-\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$
$+t(-3\overrightarrow{OA}+4\overrightarrow{OB}+5\overrightarrow{OC})-\overrightarrow{OA}$

$=(-3t-\frac{4}{3})\overrightarrow{OA}+(4t-\frac{1}{3})\overrightarrow{OB}$
$+(5t-\frac{1}{3})\overrightarrow{OC}$ //

$\therefore \overrightarrow{OH}$

$=(-3t-\frac{1}{3})\overrightarrow{OA}+(4t-\frac{1}{3})\overrightarrow{OB}$
$+(5t-\frac{1}{3})\overrightarrow{OC}$

(係数の和) $=6t-1=1$

$\therefore \underline{t=\frac{1}{3}}$ //

$\overrightarrow{DH}=\frac{1}{3}\overrightarrow{OP}$

$|\overrightarrow{DH}|^2$

$=\frac{1}{9}|-3\overrightarrow{OA}+4\overrightarrow{OB}+5\overrightarrow{OC}|^2$

$=\frac{1}{9}\{50r^2-24(r^2-5)-30(r^2-2)$
$+40(r^2-9)\}$

$=\frac{1}{9}(36r^2-180)$

$=4r^2-20$

$\therefore |\overrightarrow{DH}|=2\sqrt{r^2-5}$

(四面体ABCD)

$=3\times 2\sqrt{r^2-5}\times\frac{1}{3}=\underline{2\sqrt{r^2-5}}$ //

3.

(1) $\sin((2n+1)\pi x)=0$

$x=\frac{k}{2n+1}\quad(k=0,1,\cdots,2n+1)$

$\therefore$ T … $\underline{2n+2\text{コ}}$ //

$\underline{x_k=\frac{k}{2n+1}}$ //

(2)

$x_k \leqq x \leqq x_{k+1}$ において $f(x)$ は単調増加より

$f(x_k)\leqq f(x)\leqq f(x_{k+1})$

すべての辺に $\sin((2n+1)\pi x)$ をかけて $x_k\leqq x\leqq x_{k+1}$ で積分すると

(k が偶数のとき $\sin((2n+1)\pi x)\geqq 0$)

$\int_{x_k}^{x_{k+1}} f(x_k)\sin((2n+1)\pi x)dx$

$\leqq a_k \leqq \int_{x_k}^{x_{k+1}} f(x_{k+1})\sin((2n+1)\pi x)dx$

ここで

$\int_{x_k}^{x_{k+1}}\sin((2n+1)\pi x)dx$

$=\left[-\frac{\cos((2n+1)\pi x)}{(2n+1)\pi}\right]_{x_k=\frac{k}{2n+1}}^{x_{k+1}=\frac{k+1}{2n+1}}$

$=\frac{2}{(2n+1)\pi}$ より (F1) が示された.

(3)

$k=0\sim 2n$ まで (F1) と (F2) を適用すると

$f(x_0)\frac{2}{(2n+1)\pi}\leqq a_0\leqq f(x_1)\frac{2}{(2n+1)\pi}$

$-f(x_2)\frac{2}{(2n+1)\pi}\leqq a_1\leqq -f(x_1)\frac{2}{(2n+1)\pi}$

$f(x_2)\frac{2}{(2n+1)\pi}\leqq a_2\leqq f(x_3)\frac{2}{(2n+1)\pi}$

$-f(x_4)\frac{2}{(2n+1)\pi}\leqq a_3\leqq -f(x_3)\frac{2}{(2n+1)\pi}$

$\vdots$

$f(x_{2n})\frac{2}{(2n+1)\pi}\leqq a_{2n}\leqq f(x_{2n+1})\frac{2}{(2n+1)\pi}$

辺々たすと

$f(x_0)\frac{2}{(2n+1)\pi}\leqq\sum_{k=0}^{2n}a_k\leqq f(x_{2n+1})\frac{2}{(2n+1)\pi}$

$\therefore f(0)\frac{2}{(2n+1)\pi}\leqq\sum_{k=0}^{2n}a_k\leqq f(1)\frac{2}{(2n+1)\pi}$

すべての辺に $n\to\infty$ とすれば

$\lim_{n\to\infty}\sum_{k=0}^{2n}a_k=\lim_{n\to\infty}I_n=0$

(4)

kが奇数のときは(F2)において

$$f(x_k)\frac{2}{(2n+1)\pi} \leqq -a_k \leqq f(x_{k+1})\frac{2}{(2n+1)\pi}$$

となる.

$$\sum_{k=0}^{2n}\int_{x_k}^{x_{k+1}} f(x)|\sin((2n+1)\pi x)|dx$$

$$= a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} = J_n$$

(F1)と上の不等式を使い辺々たすと

$$\{f(x_0)+f(x_1)+\cdots+f(x_{2n})\}\frac{2}{(2n+1)\pi}$$

$$\leqq J_n \leqq \{f(x_1)+f(x_2)+\cdots+f(x_{2n+1})\}\frac{2}{(2n+1)\pi}$$

すべての辺に $n\to\infty$ とすると

$$\lim_{n\to\infty}(\text{最左辺})$$

$$= \lim_{n\to\infty}\left\{\sum_{k=1}^{2n+1} f(x_k) + f(x_0) - f(x_{2n+1})\right\}\frac{2}{(2n+1)\pi}$$

$$= \lim_{n\to\infty}\frac{2}{(2n+1)\pi}\sum_{k=1}^{2n+1} f(x_k)$$

$$= \lim_{n\to\infty}(\text{最右辺})$$

$$= \lim_{n\to\infty}\frac{2}{\pi}\cdot\frac{1}{2n+1}\sum_{k=1}^{2n+1} f\left(\frac{k}{2n+1}\right)$$

$$= \frac{2}{\pi}\int_0^1 f(x)\,dx$$

$$\therefore \lim_{n\to\infty} J_n = \frac{2}{\pi}\int_0^1 f(x)\,dx$$

4.

(1)

$$P(X_1+X_2 > X_1X_2)$$

$$= P(1 > (X_1-1)(X_2-1))$$

$$= P(X_1=1 \text{ または } X_2=1)$$

$$= P(\text{少なくとも1が1回出る})$$

$$= 1 - P(\text{1が1回も出ない})$$

$$= 1 - \left(\frac{9}{10}\right)^2 = \frac{19}{100}$$

(2)

$$P(1,2,3,4\text{が1回ずつ出る})$$

$$= 4!\cdot\frac{1}{10}\cdot\frac{2}{10}\cdot\frac{3}{10}\cdot\frac{4}{10} = \frac{36}{625}$$

$$P(\text{最大が}3)$$

$$= P(\text{すべて3以下}) - P(\text{すべて2以下})$$

$$= \left(\frac{6}{10}\right)^4 - \left(\frac{3}{10}\right)^4 = \frac{243}{2000}$$

(3)

$$P(1\text{と}k\text{の2種類})\ (k=2,3,4)$$

$$= \sum_{k=2}^{4}\frac{{}_nC_1 1\cdot k^{n-1} + {}_nC_2 1^2\cdot k^{n-2} + \cdots + {}_nC_{n-1}1^{n-1}\cdot k}{10^n}$$

$$= \sum_{k=2}^{4}\frac{(1+k)^n - 1 - k^n}{10^n}$$

$$= \frac{3^n-1-2^n+4^n-1-3^n+5^n-1-4^n}{10^n}$$

$$= \frac{5^n-2^n-3}{10^n}$$

$$n(\text{2種類})$$

$$= 5^n - 2^n + 3$$

$$+ n(2\text{と}3\text{の2種類})$$

$$+ n(2\text{と}4\text{の}\ 〃\ )$$

$$+ n(3\text{と}4\text{の}\ 〃\ )$$

$$= 5^n - 2^n + 3$$

$$+ (2+3)^n - 2^n - 3^n$$

$$+ (2+4)^n - 2^n - 4^n$$

$$+ (3+4)^n - 3^n - 4^n$$

$$= 7^n + 6^n + 2\cdot5^n - 2\cdot4^n - 2\cdot3^n - 3\cdot2^n + 3$$

$$\lim_{n\to\infty} P_n^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{5^n-2^n+3}{7^n+6^n+2\cdot5^n-2\cdot4^n-2\cdot3^n-3\cdot2^n+3}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{5}{7}\left\{\frac{1-\left(\frac{2}{5}\right)^n+\frac{3}{5^n}}{1+\left(\frac{6}{7}\right)^n+2\left(\frac{5}{7}\right)^n-2\left(\frac{4}{7}\right)^n-2\left(\frac{3}{7}\right)^n-3\left(\frac{2}{7}\right)^n+\frac{3}{7^n}}\right\}^{\frac{1}{n}}$$

$$= \frac{5}{7}\cdot 1 = \frac{5}{7}$$

(4)

1が初めて

4が少なくとも1回出る

i回目

1と4以外

上の場合は

$$\sum_{i=1}^{n-1} 5^{i-1}\cdot 1\cdot(10^{n-i} - 6^{n-i})$$

$$= \sum_{i=1}^{n-1} 5^i\cdot 5^{-1}(10^{n-i} - 6^{n-i})$$

$$= \frac{10^n}{5}\sum_{i=1}^{n-1}\left(\frac{5}{10}\right)^i - \frac{6^n}{5}\sum_{i=1}^{n-1}\left(\frac{5}{6}\right)^i$$

$$= \frac{10^n}{5}\cdot\frac{\frac{5}{10}-\left(\frac{5}{10}\right)^n}{1-\frac{5}{10}} - \frac{6^n}{5}\cdot\frac{\frac{5}{6}-\left(\frac{5}{6}\right)^n}{1-\frac{5}{6}}$$

$$= \frac{10^n}{5}\left\{1-\frac{10}{5}\left(\frac{5}{10}\right)^n\right\} - \frac{6^n}{5}\left\{5-6\left(\frac{5}{6}\right)^n\right\}$$

$$= \frac{10^n}{5} - \frac{2}{5}5^n - 6^n + \frac{6}{5}5^n$$

$$= \frac{10^n - 5\cdot6^n + 4\cdot5^n}{5}$$

全事象は10^nより求める確率は

$$\frac{10^n - 5\cdot6^n + 4\cdot5^n}{5\cdot10^n}$$

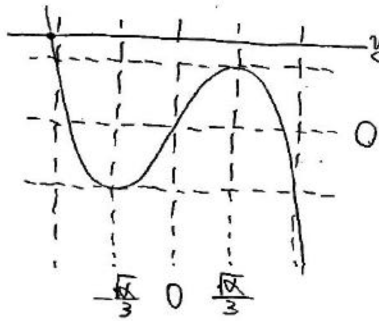
5.

(1)

$\beta = \underbrace{-18x^3+6\alpha x}_{g(x)}$

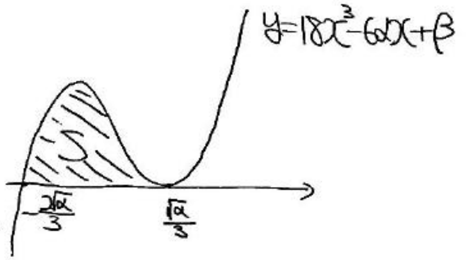
$g'(x) = -54x^2+6\alpha$

$= -6(9x^2-\alpha)$



図より

$\beta > g\left(\frac{\sqrt{\alpha}}{3}\right) = \underline{\frac{4}{3}\alpha\sqrt{\alpha}}$ //

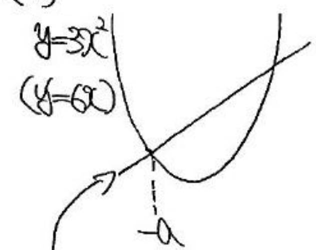


S

$= 18\int_{-\frac{2\sqrt{\alpha}}{3}}^{\frac{\sqrt{\alpha}}{3}} \left(x-\frac{\sqrt{\alpha}}{3}\right)^2\left(x+\frac{2\sqrt{\alpha}}{3}\right)dx$

$= 18\cdot\frac{1}{12}\left(\frac{\sqrt{\alpha}}{3}+\frac{2\sqrt{\alpha}}{3}\right)^4 = \underline{\frac{3}{2}\alpha^2}$ //

(2)



$y = \frac{1}{6a}(x+a)+3a^2$

$= \frac{1}{6a}x+3a^2+\frac{1}{6}$

Cと連立

$3x^2 = \frac{1}{6a}x+3a^2+\frac{1}{6}$

$\Leftrightarrow 18ax^2 = x+18a^3+a$

$\Leftrightarrow 18ax^2-x-18a^3-a=0$

$\Leftrightarrow \{18ax-(18a^2+1)\}(x+a)=0$

$\therefore x = \frac{18a^2+1}{18a},\ -a$

$\therefore X(a) = \underline{a+\frac{1}{18a}}$ //

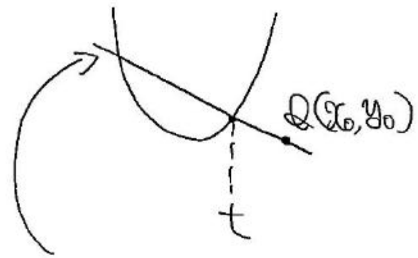
$\geqq 2\sqrt{\frac{1}{18}}$ ($\because$(相加平均) $\geqq$ (相乗平均))

$= \frac{\sqrt{2}}{3}$

$a = \frac{1}{18a} \Leftrightarrow a = \frac{\sqrt{2}}{6}$ ($\because a>0$)

のとき

$X(a)$の最小値 $\underline{\frac{\sqrt{2}}{3}}$ //



$y = -\frac{1}{6t}(x-t)+3t^2$

$\downarrow$ Q通る

$y_0 = -\frac{1}{6t}(x_0-t)+3t^2$

$\Leftrightarrow 6ty_0 = -x_0+t+18t^3$

$\Leftrightarrow 18t^3+(1-6y_0)t-x_0=0$

$\Leftrightarrow -x_0 = -18t^3+\underbrace{(6y_0-1)}_{6\alpha}t \cdots ①$

$-x_0<0$ より実数解が1つであるには

(i) $6y_0-1>0 \Leftrightarrow y_0>\frac{1}{6}$ のとき

$-x_0 < -\frac{4}{3}\alpha\sqrt{\alpha}$

$\Leftrightarrow -x_0 < -\frac{4}{3}\left(y_0-\frac{1}{6}\right)\sqrt{y_0-\frac{1}{6}}$

$\Leftrightarrow x_0 > \frac{4}{3}\left(y_0-\frac{1}{6}\right)^{\frac{3}{2}}$

$\Leftrightarrow \left(\frac{3x_0}{4}\right)^{\frac{2}{3}} > y_0-\frac{1}{6}$

$\Leftrightarrow y_0 < \left(\frac{3x_0}{4}\right)^{\frac{2}{3}}+\frac{1}{6} \cdots ②$

(ii) $6y_0-1 \leqq 0 \Leftrightarrow y_0 \leqq \frac{1}{6}$ のとき

①より$-x_0$は単調減少なもの3次関数なので，$y_0 \leqq \frac{1}{6}$のとき①はただ1つの実数解をもつ．

$\therefore y_0 \leqq \frac{1}{6}$

これは②に含まれる．

(i)(ii)より

$y_0 < \left(\frac{3x_0}{4}\right)^{\frac{2}{3}}+\frac{1}{6}$

$\therefore f(x) = \underline{\left(\frac{3x}{4}\right)^{\frac{2}{3}}+\frac{1}{6}}$ //