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[I]

(1) $\bar{x}=\dfrac{2na+nb}{3n}=\dfrac{2a+b}{3}$

$\overline{x^2}=\dfrac{2na^2+nb^2}{3n}=\dfrac{2a^2+b^2}{3}$

$s^2=\overline{x^2}-(\bar{x})^2$

$=\dfrac{2a^2+b^2}{3}-\left(\dfrac{2a+b}{3}\right)^2$

$=\dfrac{2a^2-4ab+2b^2}{9}$

$=\dfrac{2}{9}(a-b)^2$

$s=\dfrac{\sqrt{2}}{3}(b-a)$ $(\because a<b)$

$t_a=50+10\times\dfrac{a-\frac{2a+b}{3}}{s}$

$=50+10\times\dfrac{1}{3}(a-b)\times\dfrac{3}{\sqrt{2}(b-a)}$

$=\underline{50-5\sqrt{2}}$

偏差値の平均は50なので

$t_b=\underline{50+10\sqrt{2}}$

(2)

X^2-3Y^2

$=(ax+by)^2-3(cx+dy)^2$

$=(a^2-3c^2)x^2+(2ab-6cd)xy$

$+(b^2-3d^2)y^2=13$

$\downarrow$

$$\begin{cases} a^2-3c^2=1 \\ ab-3cd=0 \\ b^2-3d^2=-3 \end{cases}$$

$\downarrow$

$b^2=3d^2-3$

bは3の倍数 試しに $b=3$ を代入すると $d=2$. 2番目の式は

$a-2c=0$.

1番目の式と連立して

$a=2,\ c=1$

$\therefore (a,b,c,d)=\underline{(2,3,1,2)}$

そして $x^2=13+3y^2$

$\geqq 16$

$\therefore x\geqq 4$

より 解 $(x,y)=(4,1)$ を考えるが

$X=2x+3y=11$

$Y=x+2y=6$

より $(X,Y)=(11,6)$ は $X^2-3Y^2=13$ の解より, $(x,y)=(11,6)$ は $x^2-3y^2=13$ の解である. 同様に繰り返せば $(x,y)=(4,1),(11,6),(40,23),(149,86),(556,321)$ …と続くので求める答えは

$(x,y)=\underline{(556,321)}$

(3)

$a_n-(\alpha n+\beta)=b_n$

として,

$a_{n+2}-5a_{n+1}+6a_n-6n$

$=b_{n+2}+\{\alpha(n+2)+\beta\}-5\{b_{n+1}+\{\alpha(n+1)+\beta\}\}$

$+6\{b_n+\alpha n+\beta\}-6n$

$=b_{n+2}-5b_{n+1}+6b_n$

$+(2\alpha-6)n+(-3\alpha+2\beta)=0$

ここで $\alpha=3,\ \beta=\dfrac{9}{2}$ とすれば

b_n は普通の隣接3項間に.

$b_{n+2}-5b_{n+1}+6b_n=0$.

特 $t^2-5t+6=0$

$\therefore t=2,3$

$b_n=C_1\cdot 2^{n-1}+C_2\cdot 3^{n-1}$

とおくと

$$\begin{cases} b_1=C_1+C_2=a_1-\left(3+\frac{9}{2}\right) \\ \qquad =-\frac{13}{2} \\ b_2=2C_1+3C_2=a_2-\left(6+\frac{9}{2}\right) \\ \qquad =-\frac{19}{2} \end{cases}$$

これを解くと

$C_1=-10,\ C_2=\dfrac{7}{2}$

$\therefore b_n=-10\cdot 2^{n-1}+\dfrac{7}{2}\cdot 3^{n-1}$

$\therefore \underline{a_n=-5\cdot 2^n+\dfrac{7}{2}\cdot 3^{n-1}+3n+\dfrac{9}{2}}$

(4)

$I_n=\displaystyle\int\left\{-\frac{(\log x)^n}{x^2}\right\}dx$

$=\dfrac{1}{x}(\log x)^n-\displaystyle\int\frac{n(\log x)^{n-1}}{x^2}dx$

$=\dfrac{1}{x}(\log x)^n+nI_{n-1}$

$=\dfrac{1}{x}(\log x)^n+n\left\{\dfrac{1}{x}(\log x)^{n-1}+(n-1)I_{n-2}\right\}$

$=\dfrac{1}{x}(\log x)^n+\dfrac{n}{x}(\log x)^{n-1}$

$+n(n-1)\left\{\dfrac{1}{x}(\log x)^{n-2}+(n-2)I_{n-3}\right\}$

$$= \frac{1}{x}(\log x)^n + \frac{n}{x}(\log x)^{n-1}$$

$$+ \cdots + {}_nP_k \frac{1}{x}(\log x)^{n-k} + \cdots + \underbrace{n!\,I_0}_{{}_nP_n \cdot \frac{1}{x}}$$

$$= \sum_{k=0}^{n} \frac{{}_nP_k}{x}(\log x)^{n-k}$$

[II]

(1) カードの数字を小さい順に並べ $a_1 < a_2 < a_3$ とすると

$a_1 \to a_2 \to a_3$ …負け

$a_1 \to a_3 \to a_2$ …勝ち

$a_2 \to a_1 \to a_3$ …勝ち

$a_2 \to a_3 \to a_1$ …勝ち

$a_3 \to a_1 \to a_2$ …負け

$a_3 \to a_2 \to a_1$ …負け

となるので $P_{3,1} = \frac{1}{2}$

(2)

$$\overbrace{\overbrace{\square\cdots\square}^{k}\cdots\square\square\cdots\boxed{a_n}}^{i}\overbrace{\square\cdots\square}^{n-i}$$

最初の i−1枚の中で最大

i番目に最大のカードがきて、かつ最初の i−1枚の中で最大のものが最初の k枚の中にあればいいので求める確率は

$$\frac{1}{n}\cdot\frac{k}{i-1} = \frac{k}{n(i-1)}$$

(3)

$$P_{n,k} = \sum_{i=k+1}^{n} \frac{k}{n(i-1)}$$

$$= \frac{1}{n}\left(\frac{k}{k} + \frac{k}{k+1} + \cdots + \frac{k}{n-1}\right)$$

$$\downarrow n = 3p,\ k = p$$

$$= \frac{1}{3p}\left(\frac{p}{p} + \frac{p}{p+1} + \cdots + \frac{p}{3p-1}\right)$$

$$= \frac{1}{3p}\sum_{j=0}^{2p-1} \frac{p}{p+j}$$

$$\lim_{p\to\infty} P_{n,k}$$

$$= \lim_{p\to\infty} \frac{2}{3}\cdot\frac{1}{2p}\sum_{j=0}^{2p-1} \frac{\frac{1}{2}}{\frac{1}{2}+\frac{j}{2p}}$$

2p個での和

$$= \frac{2}{3}\int_0^1 \frac{\frac{1}{2}}{\frac{1}{2}+x}\,dx$$

$$= \frac{1}{3}\left[\log\left(\frac{1}{2}+x\right)\right]_0^1 = \frac{1}{3}\log 3$$

[III]

(1)

まず $\phi'(x)$ を求めたい。

$y = \tan x$ の逆関数の微分より

$$x = \tan y$$

を x で微分すると

$$1 = \frac{1}{\cos^2 y}\cdot\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \phi'(x) = \cos^2 y$$

$$= \frac{1}{\frac{1}{\cos^2 y}}$$

$$= \frac{1}{1+\tan^2 y}$$

$$= \frac{1}{1+x^2}$$

これより

$$f'(x)$$

$$= \frac{1}{4\sqrt{2}}\left(\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$+ \frac{1}{2\sqrt{2}}\left[\phi'(\sqrt{2}x+1)\sqrt{2} + \phi'(\sqrt{2}x-1)\sqrt{2}\right]$$

$$= \frac{1}{4}\left(\frac{\sqrt{2}x+1}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}x-1}{x^2-\sqrt{2}x+1}\right)$$

$$+ \frac{1}{2}\left[\frac{1}{1+(\sqrt{2}x+1)^2} + \frac{1}{1+(\sqrt{2}x-1)^2}\right]$$

$$= \frac{1}{4}\left(\frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}\right)$$

$$= \frac{\sqrt{2}}{4}\left(\frac{x+\sqrt{2}}{x^2+1+\sqrt{2}x} - \frac{x-\sqrt{2}}{x^2+1-\sqrt{2}x}\right)$$

$$= \frac{\sqrt{2}}{4}\cdot\frac{(x+\sqrt{2})(x^2+1-\sqrt{2}x) - (x-\sqrt{2})(x^2+1+\sqrt{2}x)}{(x^2+1)^2 - 2x^2}$$

$$= \frac{\sqrt{2}}{4}\cdot\frac{-2\sqrt{2}x^2 + 2\sqrt{2}(x^2+1)}{x^4+1}$$

$$= \frac{1}{x^4+1}$$

(2)

$$\int_0^1 \frac{1}{x^4+1}\,dx$$

$$= \left[f(x)\right]_0^1$$

$$= \frac{1}{4\sqrt{2}}\log\frac{2+\sqrt{2}}{2-\sqrt{2}} + \frac{1}{2\sqrt{2}}\left\{\phi(\sqrt{2}+1) + \phi(\sqrt{2}-1)\right\}$$

$$- \frac{1}{2\sqrt{2}}\left\{\underbrace{\phi(1)}_{\frac{\pi}{4}} + \underbrace{\phi(-1)}_{-\frac{\pi}{4}}\right\}$$

$$= \frac{1}{4\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{1}{2\sqrt{2}}\left\{\phi(\sqrt{2}+1) + \phi(\sqrt{2}-1)\right\}$$

$$= \frac{1}{4\sqrt{2}}\log(\sqrt{2}+1)^2 + \frac{\sqrt{2}}{4}\left\{\phi(\sqrt{2}+1) + \phi\left(\frac{1}{\sqrt{2}+1}\right)\right\}$$

$$= \frac{\sqrt{2}}{4}\log(\sqrt{2}+1) + \frac{\sqrt{2}}{4}\left\{\phi(\sqrt{2}+1) + \phi\left(\frac{1}{\sqrt{2}+1}\right)\right\}$$

$$\tan\theta = \frac{1}{\tan\left(\frac{\pi}{2}-\theta\right)}$$

より

$$\phi(\tan\theta) + \phi\left(\frac{1}{\tan\theta}\right) = \phi(\tan\theta) + \phi\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)$$

$$= \theta + \frac{\pi}{2} - \theta$$

$$= \frac{\pi}{2}$$

$= \frac{\sqrt{2}}{4}\log(\sqrt{2}+1) + \frac{\sqrt{2}}{4}\cdot\frac{\pi}{2}$

$\therefore\ \underline{G(x) = \frac{\sqrt{2}}{4}\left(\log x + \frac{\pi}{2}\right)}$ //

とすればいい.

(3)

$\int_{\frac{\pi}{2}}^{\pi} \frac{x\sin x}{1+\cos^4 x}dx$ 　 $\begin{array}{l} x=\pi-t \\ dx=-dt \end{array}$ 　 $\begin{array}{c|c} x & \frac{\pi}{2}\to\pi \\ \hline t & \frac{\pi}{2}\to 0 \end{array}$

$= \int_{\frac{\pi}{2}}^{0} \frac{(\pi-t)\sin t}{1+\cos^4 t}(-dt)$

$= \int_{0}^{\frac{\pi}{2}} \frac{(\pi-t)\sin t}{1+\cos^4 t}dt$

$= \int_{0}^{\frac{\pi}{2}} \frac{(\pi-x)\sin x}{1+\cos^4 x}dx$

$= \pi\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^4 x}dx - \int_{0}^{\frac{\pi}{2}} \frac{x\sin x}{1+\cos^4 x}dx$

$\Leftrightarrow \int_{0}^{\pi} \frac{x\sin x}{1+\cos^4 x}dx = \pi\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^4 x}dx$

(4)

$\int_{0}^{\pi} \frac{x\sin x}{1+\cos^4 x}dx$

$= \pi\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^4 x}dx$ 　 $\begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \end{array}$ 　 $\begin{array}{c|c} x & 0\to\frac{\pi}{2} \\ \hline t & 1\to 0 \end{array}$

$= \pi\int_{1}^{0} \frac{-1}{1+t^4}dt$

$= \pi\int_{0}^{1} \frac{1}{1+t^4}dt$

$= \pi G(\sqrt{2}+1)$

$= \frac{\sqrt{2}\pi}{4}\left\{\log(\sqrt{2}+1) + \frac{\pi}{2}\right\}$

[IV]

(1)

$(\cos\theta + i\sin\theta)^7$

$= \cos^7\theta + {}_7C_1 i\cos^6\theta\sin\theta - {}_7C_2\cos^5\theta\sin^2\theta$

$- {}_7C_3 i\cos^4\theta\sin^3\theta + {}_7C_4\cos^3\theta\sin^4\theta$

$+ {}_7C_5 i\cos^2\theta\sin^5\theta - {}_7C_6\cos\theta\sin^6\theta$

$- {}_7C_7 i\sin^7\theta$

虚部の比較すると

$\sin 7\theta$

$= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta$

$\underline{\quad + 21\cos^2\theta\sin^5\theta - \sin^7\theta}$ //

(2)

$\sin 7\theta = 0$

$\Leftrightarrow 7\theta = k\pi \quad (k\in\mathbb{Z})$

$-\pi < \theta < \pi$ とすると

$\theta = -\frac{3\pi}{7}, -\frac{2\pi}{7}, -\frac{\pi}{7}, 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}$

一方で

$\sin 7\theta$

$= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta$

$+ 21\cos^2\theta\sin^5\theta - \sin^7\theta = 0$

$\Leftrightarrow$

$\sin^7\theta\left\{7\left(\frac{1}{\tan\theta}\right)^6 - 35\left(\frac{1}{\tan\theta}\right)^4 + 21\left(\frac{1}{\tan\theta}\right)^2 - 1\right\} = 0$

$\Leftrightarrow$

$\sin^7\theta\left\{7\left(\frac{1}{\tan^2\theta}\right)^3 - 35\left(\frac{1}{\tan^2\theta}\right)^4 + 21\left(\frac{1}{\tan^2\theta}\right) - 1\right\} = 0$

これより)

$\underline{x = \frac{1}{\tan^2\theta} \quad \left(\theta = \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}\right)}$ //

は $7x^3 - 35x^2 + 21x - 1 = 0$ の解.

(3)

$\frac{1}{\tan^2\frac{\pi}{7}},\ \frac{1}{\tan^2\frac{2\pi}{7}},\ \frac{1}{\tan^2\frac{3\pi}{7}}$

は3次方程式の解より), 解と係数の関係から

$\frac{1}{\tan^2\frac{\pi}{7}} + \frac{1}{\tan^2\frac{2\pi}{7}} + \frac{1}{\tan^2\frac{3\pi}{7}} = \frac{35}{7}$

$= \underline{5}$ //