2016 慈恵医大

$\boxed{1}$

(1)

P(2回で最初と同じ)

$= P(\text{同色を選ぶ})^2$

$+ P(\text{異色を選ぶ}) \cdot P(\text{2回目が1回目と同じ})$

$= \left(\frac{2}{{}_4C_2}\right)^2 + \frac{4}{{}_4C_2} \cdot \frac{1}{{}_4C_2}$

$= \left(\frac{1}{3}\right)^2 + \frac{2}{3} \cdot \frac{1}{6} = \frac{2}{9} \cdots (ア)$

P(2回で赤赤白白)

$= P(\text{2,3を引く}) \cdot P(\text{同色を選ぶ}) \times 2$

$+ P(\text{1,2を引く}) \cdot P(\text{1,3を引く})$

$+ P(\text{3,4を引く}) \cdot P(\text{2,4を引く})$

$= \frac{1}{6} \cdot \frac{1}{3} \cdot 2 + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6}$

$= \frac{1}{6} \cdots (イ)$

(2)

z

$= \cos\theta - \sqrt{3}\sin\theta + i(\sin\theta + \sqrt{3}\cos\theta)$

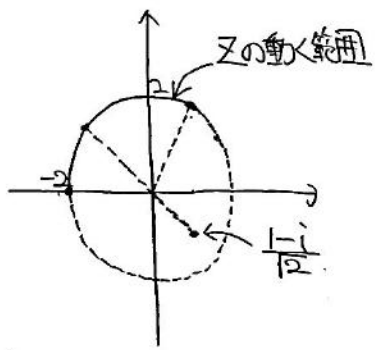
$= 2\left(\cos\theta \cdot \frac{1}{2} - \sin\theta \cdot \frac{\sqrt{3}}{2}\right) + 2i\sin\left(\theta + \frac{\pi}{3}\right)$

$= 2\left\{\cos\left(\theta + \frac{\pi}{3}\right) + i\sin\left(\theta + \frac{\pi}{3}\right)\right\}$

範囲 $\frac{\pi}{3} \leqq \theta + \frac{\pi}{3} \leqq \pi$

$|\sqrt{2}z - 1 + i|$

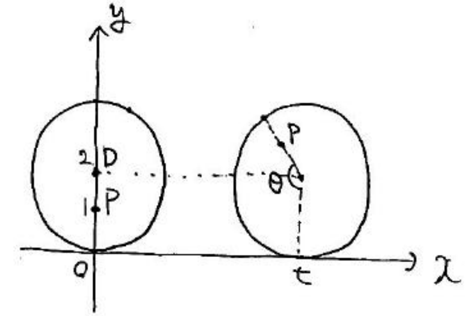
$= \sqrt{2}\left|z - \frac{1-i}{\sqrt{2}}\right|$



図より

$\max \sqrt{2}\left|z - \frac{1-i}{\sqrt{2}}\right| = 3\sqrt{2} \cdots (ウ)$

$\boxed{2}$



(1)

(i) Pの回転角をθとすると

$4\pi \times \frac{\theta}{2\pi} = t$

$\therefore \theta = \frac{t}{2}$

またPが時計まわりに回転することを考え

$\overrightarrow{DP} = \begin{pmatrix} \cos\left(\frac{3}{2}\pi - \theta\right) \\ \sin\left(\frac{3}{2}\pi - \theta\right) \end{pmatrix}$

$= \begin{pmatrix} -\sin\theta \\ -\cos\theta \end{pmatrix} = \begin{pmatrix} -\sin\frac{t}{2} \\ -\cos\frac{t}{2} \end{pmatrix}$

$\therefore \overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{DP}$

$= \begin{pmatrix} t - \sin\frac{t}{2} \\ 2 - \cos\frac{t}{2} \end{pmatrix}$

$\therefore P\left(t - \sin\frac{t}{2},\ 2 - \cos\frac{t}{2}\right) \cdots (エ)(オ)$

(ii)

$x = t - \sin\frac{t}{2} \qquad y = 2 - \cos\frac{t}{2}$

$\frac{dx}{dt} = 1 - \frac{1}{2}\cos\frac{t}{2} > 0 \qquad \frac{dy}{dt} = \frac{1}{2}\sin\frac{t}{2}$

xは単調増加であり，

yは $4n\pi \leqq t \leqq 2(2n+1)\pi$ で増加

$2(2n+1)\pi \leqq t \leqq 4(n+1)\pi$ で減少．

（nは0以上の整数）

(2)

$l : y = -\frac{1}{\frac{dy}{dx}}\left(x - t + \sin\frac{t}{2}\right) + 2 - \cos\frac{t}{2}$

$= -\frac{1 - \frac{1}{2}\cos\frac{t}{2}}{\frac{1}{2}\sin\frac{t}{2}}\left(x - t + \sin\frac{t}{2}\right) + 2 - \cos\frac{t}{2}$

$\downarrow$ $y=0$を代入

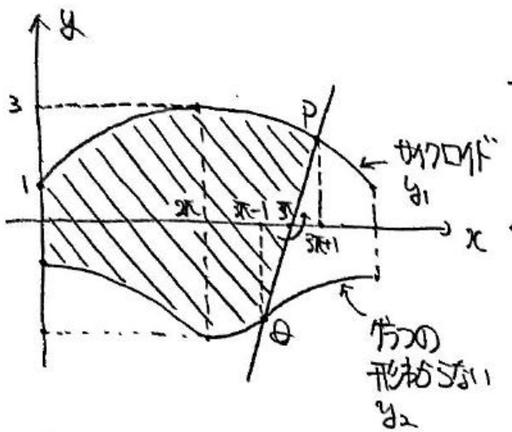
$0 = -\frac{2 - \cos\frac{t}{2}}{\sin\frac{t}{2}}\left(x - t + \sin\frac{t}{2}\right) + 2 - \cos\frac{t}{2}$

$0 = -\frac{1}{\sin\frac{t}{2}}\left(x - t + \sin\frac{t}{2}\right) + 1$

$\therefore x = t$

$\therefore M(t, 0) \leftarrow$ PQの中点

$\therefore Q\left(t + \sin\frac{t}{2},\ -2 + \cos\frac{t}{2}\right)$



(斜線部分)

$= \int_0^{3\pi+1} y_1\,dx + \int_0^{3\pi-1} (-y_2)\,dx$

$= \int_0^{3\pi} \left(2-\cos\frac{t}{2}\right)\left(1-\frac{1}{2}\cos\frac{t}{2}\right)dt$

$+ \int_0^{3\pi} \left(2-\cos\frac{t}{2}\right)\left(1+\frac{1}{2}\cos\frac{t}{2}\right)dt$

$= 2\int_0^{3\pi} \left(2-\cos\frac{t}{2}\right)dt$

$= 2\left[2t - 2\sin\frac{t}{2}\right]_0^{3\pi}$

$= \underline{12\pi + 4}$

3

(1) $a = 2m+1$ (mは自然数)

とおくと

$(2m+1)x - 2y = 1$

$-)\ (2m+1)\cdot 1 - 2m = 1$ ← 特殊解代入

$(2m+1)(x-1) - 2(y-m) = 0$

$\Leftrightarrow \underbrace{(2m+1)}_{奇数}\underbrace{(x-1)}_{\geqq 0} = 2(y-m)$

$\begin{cases} x-1 = 2k \\ y-m = (2m+1)k \end{cases}$ (kは0以上の整数)

$\therefore \begin{cases} x = 2k+1 \\ y = (2m+1)k + m \end{cases}$

よって組(x, y)は無数にある.

(2)

(与式)

$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{\frac{a}{2}x_k - \frac{1}{2}}{x_k}$

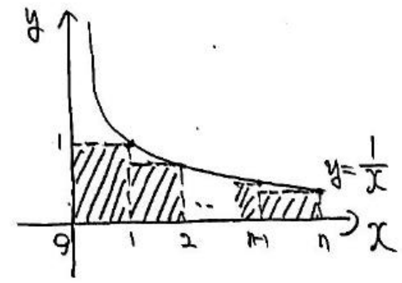
$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left(\frac{a}{2} - \frac{1}{2x_k}\right)$

$= \lim_{n\to\infty} \left(\frac{a}{2} - \frac{1}{2n}\sum_{k=1}^{n} \frac{1}{x_k}\right) \cdots ①$

ここで

$0 < \sum_{k=1}^{n} \frac{1}{x_k} < 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

($\because x_1, x_2, \cdots, x_n$は相異なる自然数)



$y = \frac{1}{x}$は単調減少より図から

$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$\leqq 1 + \int_1^n \frac{1}{x}\,dx = 1 + \log n$

よって

$0 < \frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k} \leqq \frac{1+\log n}{n}$

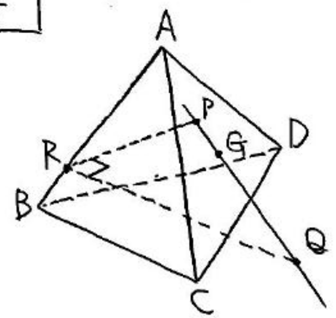
$\lim_{n\to\infty} \frac{1+\log n}{n} = 0$ より

$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k} = 0$ なので

①より

$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{y_k}{x_k} = \underline{\frac{a}{2}}$

4



$\vec{AB} = \vec{b}, \vec{AC} = \vec{c}, \vec{AD} = \vec{d}$

とおくと

$\vec{AP} = \frac{1}{8}\vec{b} + \frac{1}{4}\vec{d}$

$\vec{AG} = \frac{1}{3}\vec{c} + \frac{1}{3}\vec{d}$

より

$\vec{AQ}$

$= \vec{AP} + \vec{PQ}$

$= \vec{AP} + k\vec{PG}$

$= \vec{AP} + k\left(-\frac{1}{8}\vec{b} + \frac{1}{3}\vec{c} + \frac{1}{12}\vec{d}\right)$

$= \frac{1-k}{8}\vec{b} + \frac{k}{3}\vec{c} + \underbrace{\frac{3+k}{12}}_{0}\vec{d}$

$\therefore k = -3$

$\therefore \vec{AQ} = \frac{1}{2}\vec{b} - \vec{c}$.

また $\vec{AR} = t\vec{b}$ とおき

$\vec{PR} = \vec{AR} - \vec{AP}$

$= \left(t - \frac{1}{8}\right)\vec{b} - \frac{1}{4}\vec{d}$

$\vec{QR} = \vec{AR} - \vec{AQ}$

$= \left(t - \frac{1}{2}\right)\vec{b} + \vec{c}$

$\overrightarrow{PR}\cdot\overrightarrow{QR}$

$= \left\{\left(t-\frac{1}{8}\right)\vec{b}-\frac{1}{4}\vec{a}\right\}\left\{\left(t-\frac{1}{2}\right)\vec{b}+\vec{c}\right\}$

$= |\vec{b}|^2\left\{\left(t-\frac{1}{8}\right)\left(t-\frac{1}{2}\right)+\frac{1}{2}\left(t-\frac{1}{8}\right)\right.$

$\left.-\frac{1}{8}\left(t-\frac{1}{2}\right)-\frac{1}{8}\right\}=0$

$\Leftrightarrow$

$t^2-\frac{5}{8}t+\frac{1}{16}+\frac{4t}{8}-\frac{1}{16}$

$-\frac{t}{8}+\frac{1}{16}-\frac{1}{8}=0$

$\Leftrightarrow t^2-\frac{1}{4}t-\frac{1}{16}=0$

$\therefore t=\frac{1+\sqrt{5}}{8}$

$(\because t>0)$

$\therefore \frac{AR}{AB}=\frac{1+\sqrt{5}}{8}$