2016 北海道大・理系（前期）

$\boxed{1}$

(1)

$|w|^2$

$= w \cdot \overline{w}$

$= (z^2-2az+1)(\overline{z}^2-2a\overline{z}+1)$

$= |z|^4-2az|z|^2+z^2$
$\quad -2a|z|^2\overline{z}+4a^2|z|^2-2az$
$\quad +\overline{z}^2-2a\overline{z}+1$

$= 16-8az+z^2$
$\quad -8a\overline{z}+16a^2-2az$
$\quad +\overline{z}^2-2a\overline{z}+1$

$= z^2+\overline{z}^2-10a\underbrace{(z+\overline{z})}_{2x}+16a^2+17$

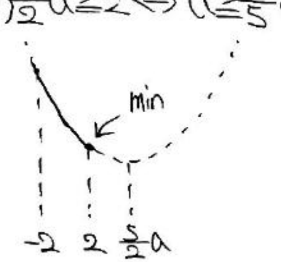
$= (z+\overline{z})^2-2|z|^2-20ax$
$\quad +16a^2+17$

$= \underline{4x^2-20ax+16a^2+9}$ //

$= 4\left(x-\frac{5}{2}a\right)^2-9a^2+9$

$= f(x)$ とおく $\quad (-2 \leqq x \leqq 2)$

(2) (i) $\frac{5}{2}a \geqq 2 \Leftrightarrow a \geqq \frac{4}{5}$ のとき



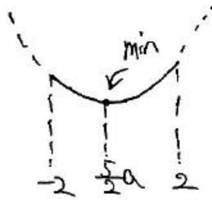
$\min |w|^2 = f(2)$

$= 16a^2-40a+25$

$= (4a-5)^2$

$\therefore (|w|の最小値) = \underline{|4a-5|}$ //

(ii) $-2 \leqq \frac{5}{2}a \leqq 2 \Leftrightarrow -\frac{4}{5} \leqq a \leqq \frac{4}{5}$ のとき

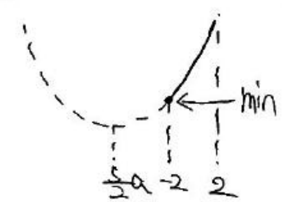


$\min |w|^2 = f\left(\frac{5}{2}a\right)$

$= -9a^2+9$

$\therefore (|w|の最小値) = \underline{3\sqrt{-a^2+1}}$ //

(iii) $\frac{5}{2}a \leqq -2 \Leftrightarrow a \leqq -\frac{4}{5}$ のとき



$\min |w|^2 = f(-2) = 16a^2+40a+25$

$\therefore (|w|の最小値) = |4a+5|$ //

$\boxed{2}$

(1)

$f(x)$

$= \left[\frac{e^{-x}}{2a}t\right]_{-a}^{a} + \int_{-a}^{a} f(t)\sin t\,dt$

$= e^{-x}+b$ とおく

$b = \int_{-a}^{a}(e^{-t}+b)\sin t\,dt$

$= \int_{-a}^{a} e^{-t}\sin t\,dt + b\underbrace{\int_{-a}^{a}\sin t\,dt}_{0}$

$= \left[-e^{-t}\sin t\right]_{-a}^{a} - \int_{-a}^{a}(-e^{-t}\cos t)\,dt$

$= -e^{-a}\sin a + e^{a}\sin(-a)$
$\quad + \int_{-a}^{a} e^{-t}\cos t\,dt$

$= -\sin a(e^{a}+e^{-a})$
$\quad + \left[-e^{-t}\cos t\right]_{-a}^{a} - \underbrace{\int_{-a}^{a} e^{-t}\sin t\,dt}_{b}$

$2b = -(e^{a}+e^{-a})\sin a$
$\quad -e^{-a}\cos a + e^{a}\cos(-a)$

$\therefore b = \dfrac{e^{a}(\cos a-\sin a)-e^{-a}(\cos a+\sin a)}{2}$

$\therefore f(x)$

$= \underline{e^{-x}+\dfrac{e^{a}(\cos a-\sin a)-e^{-a}(\cos a+\sin a)}{2}}$ //

(2)

$b = g(a)$

$g'(a)$

$= \frac{1}{2}\{e^{a}(\cos a-\sin a)+e^{a}(-\sin a-\cos a)$
$\quad +e^{-a}(\cos a+\sin a)-e^{-a}(-\sin a+\cos a)\}$

$= -e^{a}\sin a+e^{-a}\sin a$

$= (e^{-a}-e^{a})\sin a$

a	0	$\cdots$	π	$\cdots$	2π
$g'(a)$	0	$-$	0	$+$	0
$g(a)$	0	$\searrow$		$\nearrow$	

$a=\pi$ のとき 最小値 $\underline{\dfrac{e^{-\pi}-e^{\pi}}{2}}$ //

$\boxed{3}$

(1)

P(Aが2種類)

$= 1-P$(Aが1種類)$-P$(Aが3種類)

$= 1-\frac{3}{3^3}-\frac{3!}{3^3}$

$= \underline{\frac{2}{3}}$ //

(2)

$P(\text{A.Bあわせて2種類})$

$= P(\text{金と銀}) + P(\text{銀と銅})$

$\quad + P(\text{銅と金})$

$= \dfrac{2^5-2}{3^3 \cdot 2^2} + \dfrac{(2^3-1)\cdot 1^2}{3^3 \cdot 2^2} \times 2$

$= \dfrac{44}{3^3 \cdot 2^2} = \underline{\dfrac{11}{27}}$ //

(3)

X…A.Bあわせて金が3枚.

Y…Aが2種類

とおく.

$P(X)$

$= P(A:\text{金}\times 3\ \ B:\text{金}\times 0)$

$+ P(A:\text{金}\times 2,\ B:\text{金}\times 1)$

$+ P(A:\text{金}\times 1,\ B:\text{金}\times 2)$

$= \dfrac{1}{3^3} \cdot \dfrac{1}{2^2} + \dfrac{3 \cdot 2}{3^3} \cdot \dfrac{2}{2^2}$

$+ \dfrac{3! + 3 \cdot 2}{3^3} \cdot \dfrac{1}{2^2}$

$= \dfrac{1+12+12}{108} = \dfrac{25}{108}$

$P(X \cap Y)$

$= \dfrac{3 \cdot 2}{3^3} \cdot \dfrac{2}{2^2} + \dfrac{3 \cdot 2}{3^3} \cdot \dfrac{1}{2^2}$

$= \dfrac{18}{108}$

$\therefore P_X(Y) = \dfrac{P(X \cap Y)}{P(X)} = \underline{\dfrac{18}{25}}$ //

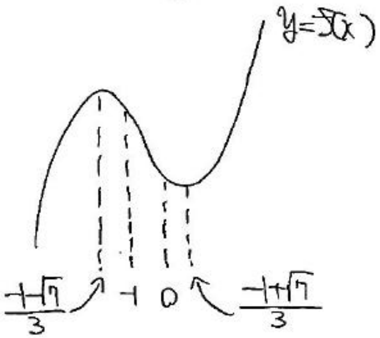
$\boxed{4}$

(1)

$f(x) = x^3 + x^2 - 2x - 1$ とおく

$f'(x) = 3x^2 + 2x - 2 = 0$

$x = \dfrac{-1 \pm \sqrt{7}}{3}$



x	…	$\frac{-1-\sqrt{7}}{3}$	…	-1	…	0	…	$\frac{-1+\sqrt{7}}{3}$	…
$f'(x)$	+	0	−	−	−	−	−	0	+
$f(x)$	↗		↘	1	↘	−1	↘		↗

$\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$

これらより $f(x)=0$ は

$x < \dfrac{-1-\sqrt{7}}{3},\ -1 < x < 0,\ \dfrac{-1+\sqrt{7}}{3} < x$

に1つずつ実数解をもつ.

(2)

$a_{n+3} + a_{n+2} - 2a_{n+1} - a_n$

$= \dfrac{s^{n-1}}{(s-t)(s-u)} \underbrace{(s^3+s^2-2s-1)}_{0}$

$+ \dfrac{t^{n-1}}{(t-u)(t-s)} \underbrace{(t^3+t^2-2t-1)}_{0}$

$+ \dfrac{u^{n-1}}{(u-s)(u-t)} \underbrace{(u^3+u^2-2u-1)}_{0}$

$= 0$

(3) (i) $n=1,2,3$ のとき

a_1

$= \dfrac{1}{(s-t)(s-u)} + \dfrac{1}{(t-u)(t-s)} + \dfrac{1}{(u-s)(u-t)}$

$= \dfrac{u-t-u+s+t-s}{(s-t)(t-u)(u-s)} = 0$

a_2

$= \dfrac{s(u-t)+t(-u+s)+u(t-s)}{(s-t)(t-u)(u-s)} = 0$

a_3

$= \dfrac{s^2(u-t)+t^2(-u+s)+u^2(t-s)}{(s-t)(t-u)(u-s)}$

$(\text{分子}) = (u-t)s^2 - (u^2-t^2)s + ut(u-t)$

$= (u-t)(s^2-(u+t)s+ut)$

$= (u-t)(s-u)(s-t)$

より

$a_3 = 1$ なのですべて整数

(ii) $n = l, l+1, l+2\ (l \geqq 1)$ のとき

整数であると仮定すると

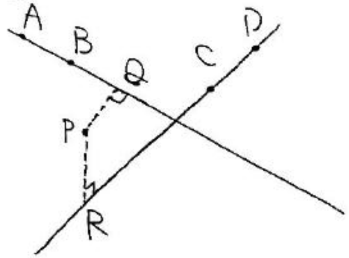
$a_{l+3} = -a_{l+2} + 2a_{l+1} + a_l$

より $n = l+3$ ときも整数になる.

(i)(ii)よりすべての自然数 n において

a_n は整数.

5



(1) lの媒介変数表示

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overrightarrow{OA} + k\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + k\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ k \\ k+2 \end{pmatrix}$$

$Q(0,\ q,\ q+2)$とおくと

$$\overrightarrow{AB} \cdot \overrightarrow{PQ} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -s \\ q-t \\ q+2-a \end{pmatrix} = q-t+q+2-a=0$$

$$\therefore q = \frac{t+a-2}{2}$$

$$\therefore Q\left(0,\ \frac{t+a-2}{2},\ \frac{t+a+2}{2}\right)$$

mの媒介変数表示

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overrightarrow{OC} + k\overrightarrow{CD} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix}$$

$R(1,\ 0,\ r)$とおくと

$$\overrightarrow{CD} \cdot \overrightarrow{PR} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1-s \\ -t \\ r-a \end{pmatrix} = r-a=0$$

$$\therefore r=a$$

$$\therefore R(1,\ 0,\ a)$$

(2)

$|\overrightarrow{PQ}| = |\overrightarrow{PR}|$

であればよいので2乗すると

$$s^2+(q-t)^2+(q+2-a)^2=(1-s)^2+t^2$$

$$\Leftrightarrow s^2+\left(\frac{t+a-2}{2}\right)^2+\left(\frac{t-a+2}{2}\right)^2=(1-s)^2+t^2$$

$$\Leftrightarrow s^2+\frac{\{t-(a-2)\}^2}{2}=1-2s+s^2+t^2$$

$$\Leftrightarrow \frac{t^2-2(a-2)t+(a-2)^2}{2}=1-2s+t^2$$

$$\Leftrightarrow -2(a-2)t+(a-2)^2=2-4s+t^2$$

$$\Leftrightarrow 4s=t^2+2(a-2)t-a^2+4a-2$$

(3)

$$4s=\{t+(a-2)\}^2-(a-2)^2-a^2+4a-2$$

$$\Leftrightarrow \{t+(a-2)\}^2=4s+2a^2-8a+6$$

$$\Leftrightarrow \{t-(2-a)\}^2=4\left(s+\frac{a^2-4a+3}{2}\right)$$

これより $\left(s+\frac{a^2-4a+3}{2},\ t-(2-a)\right)$

の軌跡は焦点が$(1,0)$, 準線が $s+\frac{a^2-4a+3}{2}=-1$ の放物線なので

(s,t)の軌跡は

焦点 $\left(\frac{-a^2+4a-1}{2},\ 2-a\right)$

準線 $x=\frac{-a^2+4a-5}{2}$

の放物線となる.