2015 東京医科歯科大（医）

$\boxed{1}$

(1)

$P_3(2)$

1 2 3
1 2 3

$$= \frac{{}_3C_2 \cdot 2 \cdot 2}{{}_6C_2}$$

$$= \frac{4}{5}$$

$P_3(3)$

$$= \frac{2^3}{{}_6C_3}$$

$$= \frac{2}{5}$$

$P_3(4)$

$$= 0$$

(2)

$E_{10}(m)$

$$= mP_{10}(m)$$

$$= m \cdot \frac{{}_{10}C_m \cdot 2^m}{{}_{20}C_m}$$

$$= m \cdot 2^m \frac{\frac{10!}{m!(10-m)!}}{\frac{20!}{m!(20-m)!}}$$

$$= m \cdot 2^m \frac{10!(20-m)!}{20!(10-m)!}$$

$$\frac{E_{10}(m+1)}{E_{10}(m)}$$

$$= \frac{m+1}{m} \cdot 2 \cdot \frac{\frac{(19-m)!}{(9-m)!}}{\frac{(20-m)!}{(10-m)!}}$$

$$= \frac{2(m+1)}{m} \cdot \frac{10-m}{20-m} > 1$$

$$\Leftrightarrow 2(m+1)(10-m) > m(20-m)$$

$$\Leftrightarrow 0 > m^2+2m-20$$

$$\Leftrightarrow -1-\sqrt{21} < m < -1+\sqrt{21}$$

これより

$1 \leqq m \leqq 3$ のとき $E_{10}(m) < E_{10}(m+1)$

$m \geqq 4$ のとき $E_{10}(m) > E_{10}(m+1)$

なので

$E_{10}(1) < E_{10}(2) < E_{10}(3)$

$< E_{10}(4) > E_{10}(5) > \cdots$

求める答えは $m = 4$

(3) (2) と同様に

$E_n(m)$

$$= m \cdot \frac{{}_nC_m \cdot 2^m}{{}_{2n}C_m}$$

$$= m \cdot 2^m \cdot \frac{n!(2n-m)!}{(2n)!(n-m)!}$$

より

$$\frac{E_n(m+1)}{E_n(m)}$$

$$= \frac{m+1}{m} \cdot 2 \cdot \frac{n-m}{2n-m} < 1$$

$$\Leftrightarrow 2(m+1)(n-m) < m(2n-m)$$

$$\Leftrightarrow 2(-m^2+nm-m+n) < -m^2+2nm$$

$$\Leftrightarrow 0 < m^2+2m-2n$$

$$\Leftrightarrow m < -1-\sqrt{1+2n},\ -1+\sqrt{1+2n} < m$$

m は自然数より

$$f(n) = [-1+\sqrt{1+2n}]+1$$

$$= -1+[\sqrt{1+2n}]+1$$

$$= [\sqrt{1+2n}]$$

$\boxed{2}$

$f(x) = x^3-3ax+b$

$f'(x) = 3x^2-3a$

(1)

$f'(x) = 3(x+\sqrt{a})(x-\sqrt{a})$

x	$\cdots$	$-\sqrt{a}$	$\cdots$	$\sqrt{a}$	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$

極大値：$f(-\sqrt{a})$

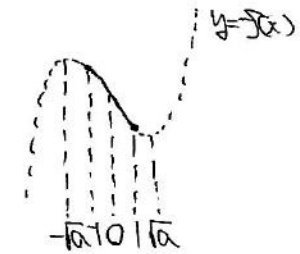
$$= 2a\sqrt{a}+b$$

極小値：$f(\sqrt{a})$

$$= -2a\sqrt{a}+b$$

(2)

(i) $a \geqq 1$ のとき

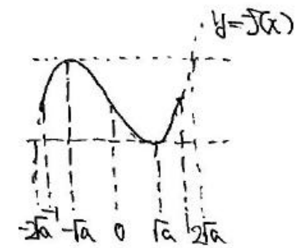


$M = f(-1)$

$= 3a+b-1$

(ii) $\sqrt{a} \leqq 1 \leqq 2\sqrt{a}$

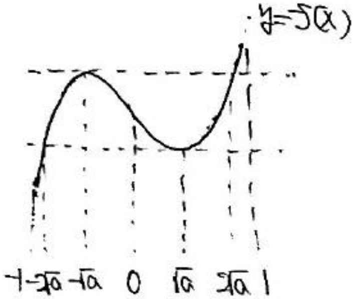
$\Leftrightarrow \frac{1}{4} \leqq a \leqq 1$ のとき



$M = f(-\sqrt{a})$

$= 2a\sqrt{a}+b$

(iii) $0 < a \leqq \frac{1}{4}$ のとき

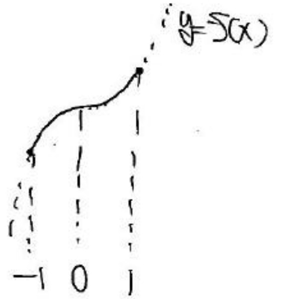


$M = f(1)$

$= -3a+b+1$

(iv) $a \leqq 0$ のとき

$f'(x) = 3x^2 - 3a \geqq 0$



$M = f(1)$

$= -3a+b+1$

以上より

$$M = \begin{cases} 3a+b-1 & (a \geqq 1) \\ 2a\sqrt{a}+b & (\frac{1}{4} \leqq a \leqq 1) \\ -3a+b+1 & (a \leqq \frac{1}{4}) \end{cases}$$

(3)

(2)より $b \geqq 0$ のとき

$M \geqq b + \frac{1}{4} \geqq \frac{1}{4}$ …①

等号成立は $a = \frac{1}{4}$, $b=0$

(2)と同様に $b<0$ のとき

$$M = \begin{cases} -f(1) & (a \geqq 1) \\ -f(\sqrt{a}) & (\frac{1}{4} \leqq a \leqq 1) \\ -f(-1) & (a \leqq \frac{1}{4}) \end{cases}$$

$$= \begin{cases} 3a-b-1 & (a \geqq 1) \\ 2a\sqrt{a}-b & (\frac{1}{4} \leqq a \leqq 1) \\ -3a-b+1 & (a \leqq \frac{1}{4}) \end{cases}$$

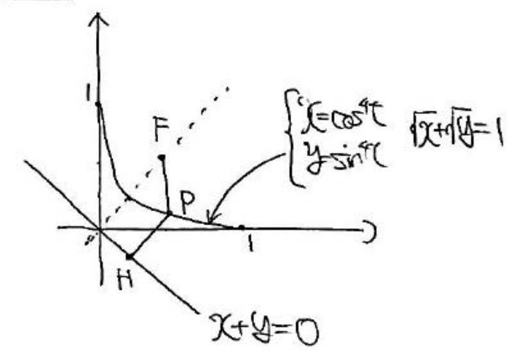
以上より

$M \geqq \frac{1}{4} - b > \frac{1}{4}$ …②

①, ②より

$M \geqq \frac{1}{4}$

3



C も $x+y=0$ も $y=x$ に関して対称なので F は $y=x$ 上にある.

$P(\cos^4\theta, \sin^4\theta)$, $F(a, a)$ とおくと

$PF^2 = PH^2$

$(a-\cos^4\theta)^2 + (a-\sin^4\theta)^2 = \frac{1}{2}(\cos^4\theta+\sin^4\theta)^2$

$2a^2 - 2(\cos^4\theta+\sin^4\theta)a + \cos^8\theta + \sin^8\theta - \frac{1}{2}(\cos^4\theta+\sin^4\theta)^2 = 0$

$2a^2 - 2(\cos^4\theta+\sin^4\theta)a + \frac{1}{2}(\cos^4\theta-\sin^4\theta)^2 = 0$

$4a^2 - 4(1-2\cos^2\theta\sin^2\theta)a + (\cos^2\theta-\sin^2\theta)^2 = 0$

$4a^2 - 4(1-2\cos^2\theta\sin^2\theta)a + (\cos^2\theta+\sin^2\theta)^2 - 4\cos^2\theta\sin^2\theta = 0$

$4a^2 - 4(1-2\cos^2\theta\sin^2\theta)a + 1 - 4\cos^2\theta\sin^2\theta = 0$

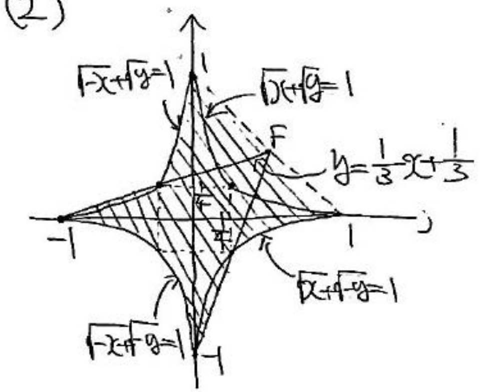
$(2a-1)\{2a-(1-4\cos^2\theta\sin^2\theta)\} = 0$

θ の値に関わらず成立つので

$a = \frac{1}{2}$

$\therefore F\left(\frac{1}{2}, \frac{1}{2}\right)$

(2)



求める面積は図の斜線部分、

$1 \cdot 1 \cdot \frac{1}{2} + \int_0^1 (1-\sqrt{x})^2 dx + 2\left\{\frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} + \int_0^{\frac{1}{4}} (1-\sqrt{x})^2 dx\right\}$

$= \frac{1}{2} + \left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2\right]_0^1 + 2\left(\frac{3}{32} + \left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2\right]_0^{\frac{1}{4}}\right)$

$= \cdots = \frac{13}{12}$

(3)

$$1 \cdot 1 \cdot \pi \cdot 1 \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{4} \pi \cdot \frac{3}{4} \cdot \frac{1}{3}$$

$$+ \int_0^{\frac{1}{4}} (1-\sqrt{x})^4 \pi \, dx \quad \sqrt{x} = t$$

$$= \frac{\pi}{3} + \frac{\pi}{64} + \int_0^{\frac{1}{2}} (1-t)^4 \pi \, 2t \, dt$$

$$= \frac{\pi}{3} + \frac{\pi}{64} \quad 1-t = u$$

$$+ \int_1^{\frac{1}{2}} u^4 \pi \cdot 2(1-u)(-du)$$

$$= \frac{\pi}{3} + \frac{\pi}{64}$$

$$+ 2\pi \int_{\frac{1}{2}}^1 (u^4 - u^5) du$$

$$= \frac{\pi}{3} + \frac{\pi}{64} + 2\pi \left[\frac{1}{5} u^5 - \frac{1}{6} u^6 \right]_{\frac{1}{2}}^1$$

$$= \frac{\pi}{3} + \frac{\pi}{64} + \frac{19}{320} \pi$$

$$= \frac{49}{120} \pi$$

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