2014 東京医科大

$\boxed{1}$

(1)

$(C_1 \text{のAでの接線の傾き}) = \frac{1}{2}$

C_2がAを通るので

$$a + \frac{b}{16} = 1.$$

C_2のAでの接線は

$$a \cdot 1 \cdot x + b \cdot \frac{1}{4} \cdot y = 1$$

$$\Leftrightarrow \left(1 - \frac{b}{16}\right)x + \frac{b}{4}y - 1 = 0$$

(C_1の接線の方向ベクトル)

// (C_2の接線の法線ベクトル)

$$\therefore \begin{pmatrix} 2 \\ 1 \end{pmatrix} // \begin{pmatrix} 1 - \frac{b}{16} \\ \frac{b}{4} \end{pmatrix}$$

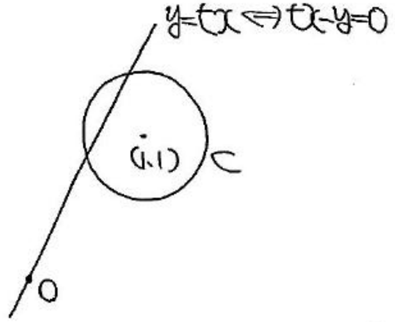
$$\therefore 2 \cdot \frac{b}{4} - 1 \cdot \left(1 - \frac{b}{16}\right) = 0$$

$$\therefore b = \frac{16}{9} \quad a = \frac{8}{9}$$

(2)

$\frac{y}{x} = t$ とおく.

C上でのtの範囲を求める.



$tx - y = 0$ と $(1,1)$ のキョリが $\frac{1}{4}$ 以下.

$$\frac{|t-1|}{\sqrt{t^2+1}} \leqq \frac{1}{4}$$

↓2乗

$$(t-1)^2 \leqq \frac{1}{16}(t^2+1)$$

$$\Leftrightarrow 15t^2 - 32t + 15 \leqq 0$$

$$\Leftrightarrow 15t^2 + 15 \leqq 32t$$

$$\Leftrightarrow t + \frac{1}{t} \leqq \frac{32}{15}$$

図より明らかに $t > 0$

$$\therefore \frac{x}{y} + \frac{y}{x} = \frac{1}{t} + t \leqq \frac{32}{15}$$

$$\therefore M = \frac{32}{15}$$

$\boxed{2}$

(1)

$\vec{p} \cdot \vec{q}$

$$= 9\cos t\cos\left(t + \frac{\pi}{3}\right) + 4\sin t\sin\left(t + \frac{\pi}{3}\right)$$

$$= 5\cos t\cos\left(t + \frac{\pi}{3}\right)$$

$$+ 4\left\{\cos t\cos\left(t + \frac{\pi}{3}\right) + \sin t\sin\left(t + \frac{\pi}{3}\right)\right\}$$

$$= 5\cos t\cos\left(t + \frac{\pi}{3}\right)$$

$$+ 4\cos\left(t - \left(t + \frac{\pi}{3}\right)\right)$$

$$= 5\cos t\cos\left(t + \frac{\pi}{3}\right) + 2$$

ここで

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$+)\ \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

$$\therefore \cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$$

$$= \frac{5}{2}\cos\left(2t + \frac{\pi}{3}\right) + \frac{5}{2}\cos\left(-\frac{\pi}{3}\right) + 2$$

$$= \frac{5}{2}\cos\left(2t + \frac{\pi}{3}\right) + \frac{13}{4}$$

$$\therefore M = \frac{5}{2} + \frac{13}{4} = \frac{23}{4}$$

$$m = -\frac{5}{2} + \frac{13}{4} = \frac{3}{4}$$

(2)

α

$$= \lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \frac{1}{n^5} \sum_{k=1}^{n} k(k+1)(k+2)(k+3)$$

$$= \lim_{n\to\infty} \frac{1}{n^5} \cdot \frac{1}{5} n(n+1)(n+2)(n+3)(n+4)$$

$$= \frac{1}{5}$$

※区分求積の方がいいね.

$$\sum_{k=1}^{n} \{(k+1)^5 - k^5\} = (n+1)^5 - 1$$

$$\sum_{k=1}^{n} \{(k+1)^5 - k^5\}$$

$$= \sum_{k=1}^{n} (5k^4 + 10k^3 + 10k^2 + 5k + 1)$$

$$\Leftrightarrow \sum_{k=1}^{n} \left(k^4 + 2k^3 + 2k^2 + k + \frac{1}{5}\right) = \frac{1}{5}\{(n+1)^5 - 1\}$$

$$\Leftrightarrow \sum_{k=1}^{n} k^4 = \frac{1}{5}\{(n+1)^5 - 1\}$$

$$- \sum_{k=1}^{n} \left(2k^3 + 2k^2 + k + \frac{1}{5}\right)$$

β

$$= \lim_{n\to\infty} \left\{\frac{n}{5} - \frac{1}{5n^4}\{(n+1)^5 - 1\}\right.$$

$$\left. + \frac{1}{n^4} \sum_{k=1}^{n} \left(2k^3 + 2k^2 + k + \frac{1}{5}\right)\right\}$$

$$= -1 + 2\left(\frac{1}{2}\right)^2 = \frac{-1}{2}$$

$\boxed{3}$

(1)

$y=\sqrt{x^2+9}$

↓2乗

$y^2=x^2+9$

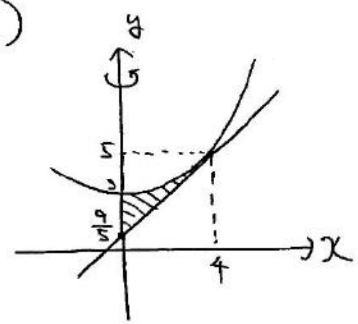
$\Leftrightarrow x^2-y^2=-9$

A(4,5)での接線は

$4x-5y=-9$

$\therefore \underline{y=\frac{4}{5}x+\frac{9}{5}}$

(2)



V

$=16\pi\cdot\frac{16}{5}\cdot\frac{1}{3}-\int_3^5 x^2\pi\,dy$

$=\frac{256}{15}\pi-\pi\int_3^5(y^2-9)\,dy$

$=\frac{256}{15}\pi-\pi\left[\frac{1}{3}y^3-9y\right]_3^5$

$=\frac{256}{15}\pi-\pi\left\{\frac{125}{3}-45-(9-27)\right\}$

$=\frac{256}{15}\pi-\frac{625}{15}\pi+27\pi=\underline{\frac{12}{5}\pi}$

$\boxed{4}$

(1)

C_1とC_2は逆関数の関係なので

C_1が$y=x$と交わる範囲を考える.

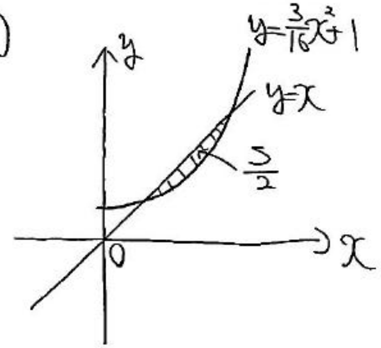
$x=ax^2+1$

$\Leftrightarrow ax^2-x+1=0$

$D=1-4a>0$

$\therefore \underline{0<a<\frac{1}{4}}$

(2)



交点は

$\frac{3x^2}{16}+1=x$

$\Leftrightarrow \frac{1}{16}(3x^2-16x+16)=0$

$\Leftrightarrow \frac{1}{16}(3x-4)(x-4)=0$

$x=\frac{4}{3},\ 4$

$\frac{S}{2}$

$=\int_{\frac{4}{3}}^4\left\{x-\left(\frac{3}{16}x^2+1\right)\right\}dx$

$=-\frac{3}{16}\int_{\frac{4}{3}}^4\left(x-\frac{4}{3}\right)(x-4)\,dx$

$=-\frac{3}{16}\left\{-\frac{1}{6}\left(4-\frac{4}{3}\right)^3\right\}$

$=\frac{1}{32}\cdot 64\left(1-\frac{1}{3}\right)^3=\frac{16}{27}$

$\therefore \underline{S=\frac{32}{27}}$