2014 慈恵医大

1.

(1) $P(Y=2X)$

$= P(X=2)+P(X=3)$
$\quad +P(X=4)+P(X=5)$

$= \dfrac{1+2+3+4}{{}_{10}C_3}$

$= \dfrac{10}{120}$

$= \dfrac{1}{12}$ //

$P(Y<2X)$

$= P(X=3)+P(X=4)$
$\quad +P(X=5)+P(X=6)$
$\quad +P(X=7)+P(X=8)$

$= \dfrac{1+{}_3C_2+{}_4C_2+{}_4C_2+{}_3C_2+1}{{}_{10}C_3}$

$= \dfrac{20}{120} = \dfrac{1}{6}$ //

(2) $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $Q(x_q, y_q)$

とおくと

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-b \\ c-d \end{pmatrix} = \begin{pmatrix} x_q \\ y_q \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_q \\ y_q \end{pmatrix} = \begin{pmatrix} ax_q+by_q \\ cx_q+dy_q \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -a \\ -c \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

↓ $a=-1,\ c=1$

$-1-b=x_q$
$1-d=y_q$
$-x_q+by_q=-1$
$x_q+dy_q=0$

↓

$-x_q+(-1-x_q)y_q=-1$
$x_q+(1-y_q)y_q=0$

↓

$-y_q(y_q-1)+\{-1-y_q(y_q-1)\}y_q=-1$

$\Leftrightarrow -y_q^3+y_q-y_q=-1$

$\therefore y_q=1,\ x_q=0$

$b=-1,\ d=0$

$\therefore A=\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ $Q(0, 1)$ //

2.

(1)

(i)

$f(x)=\log_2(a^2+x^2)$

$f'(x)=\dfrac{2x}{a^2+x^2}$

$f''(x)=2\dfrac{a^2+x^2-x\cdot 2x}{(a^2+x^2)^2}$

$=2\dfrac{a^2-x^2}{(a^2+x^2)^2}$

C_1は $0\leqq x\leqq a$ で下に凸.
$x\geqq a$ で上に凸.

$t(a)=a$ //

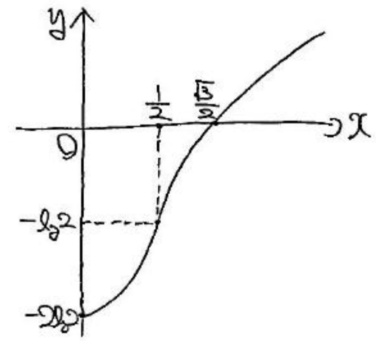
$P(a, \log_2 2a^2)$ を C_2 が通るので

$\log_2 2a^2=a^2+b$

$\therefore b=\log_2 2a^2-a^2$ //

(ii)

x	0	$\cdots$	$\frac{1}{2}$	$\cdots$
$f'(x)$	0	$+$	$+$	$+$
$f''(x)$	$+$	$+$	0	$-$
$f(x)$	$-2\log_2 2$	↗	$-\log_2 2$	↗



(2)

$h(x)=f(x)-g(x)$
$\quad =\log_2(a^2+x^2)-x^2-b$ $(0\leqq x\leqq a)$ とおく.

$h'(x)=\dfrac{2x}{a^2+x^2}-2x$

$=2x\cdot\dfrac{1-a^2-x^2}{a^2+x^2}$

$\geqq 2x\cdot\dfrac{1-a^2-a^2}{a^2+x^2}$

$\geqq 0$ $\left(\because 0<a\leqq \frac{\sqrt{2}}{2}\right)$

より $h(x)$ は単調増加.

$h(a)=f(a)-g(a)=0$ より

$h(x)\leqq 0$.

$\therefore f(x)\leqq g(x)$ //

(3)

$S(a)$

$=\int_0^a \{g(x)-f(x)\}dx$

$=\int_0^a \{x^2+b-(x)\log(a^2+x^2)\}dx$

$=\frac{a^3}{3}+ba-\left[x\log(a^2+x^2)\right]_0^a$

$+\int_0^a x\cdot\frac{2x}{a^2+x^2}dx$

$=\frac{a^3}{3}+ba-a\log 2a^2$

$+2\int_0^a\left(1-\frac{a^2}{a^2+x^2}\right)dx$

$=-\frac{2}{3}a^3+2a-2a^2\int_0^a\frac{1}{a^2+x^2}dx$

$\quad x=a\tan\theta$

$=-\frac{2}{3}a^3+2a-2a^2\int_0^{\frac{\pi}{4}}\frac{1}{\frac{a^2}{\cos^2\theta}}\cdot\frac{a}{\cos^2\theta}d\theta$

$=-\frac{2}{3}a^3+2a-2a\frac{\pi}{4}$

$=-\frac{2}{3}a^3+\frac{4-\pi}{2}a$ //

$S'(a)$

$=-2a^2+\frac{4-\pi}{2}$

$=-2\left(a^2-\frac{4-\pi}{4}\right)$

$S'(a)=0 \Leftrightarrow a=\frac{\sqrt{4-\pi}}{2}<\frac{\sqrt{2}}{2}$

a	0	$\cdots$	$\frac{\sqrt{4-\pi}}{2}$	$\cdots$	$\frac{\sqrt{2}}{2}$
$S'(a)$		$+$	0	$-$	$-$
$S(a)$		↗		↘	

$a=\frac{\sqrt{4-\pi}}{2}$ のとき最大値は //

$S\left(\frac{\sqrt{4-\pi}}{2}\right)$

$=-\frac{2}{3}\cdot\frac{(4-\pi)^{\frac{3}{2}}}{8}+\frac{1}{4}(4-\pi)^{\frac{3}{2}}$

$=\frac{1}{6}(4-\pi)^{\frac{3}{2}}$ //

3.

(1)

$\sin x+\cos x=t$ とおくと

$t=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ より

$-\sqrt{2}\leqq t\leqq\sqrt{2}$.

与式は

$-\sqrt{2}at+2b(t^2-1)-4\leqq 0$

$\Leftrightarrow 2bt^2-\sqrt{2}at-2b-4\leqq 0$

となる. 左辺を $f(t)$ とおくと

$-\sqrt{2}\leqq t\leqq\sqrt{2}$ においての $f(t)$ の

最大値が0以下であればいい.

(i) $b\geqq 0$ のとき

求める条件は

$\begin{cases} f(\sqrt{2})=-2a+2b-4\leqq 0 \text{ かつ} \\ f(-\sqrt{2})=2a+2b-4\leqq 0 \end{cases}$

$\Leftrightarrow \begin{cases} b\leqq a+2 \\ \text{かつ} \\ b\leqq -a+2 \end{cases}$

(ii) $b<0$ のとき

$f(t)=2b\left(t-\frac{\sqrt{2}a}{4b}\right)^2-\frac{a^2}{4b}-2b-4$

求める条件は

$-\sqrt{2}\leqq\frac{\sqrt{2}a}{4b}\leqq\sqrt{2}$ のとき

$\begin{cases} f(\sqrt{2})\leqq 0 \\ \text{かつ} \\ f(-\sqrt{2})\leqq 0 \\ \text{かつ} \\ -\frac{a^2}{4b}-2b-4\leqq 0 \end{cases}$

$\frac{\sqrt{2}a}{4b}<-\sqrt{2}$, $\sqrt{2}<\frac{\sqrt{2}a}{4b}$ のとき

$\begin{cases} f(\sqrt{2})\leqq 0 \\ \text{かつ} \\ f(-\sqrt{2})\leqq 0 \end{cases}$

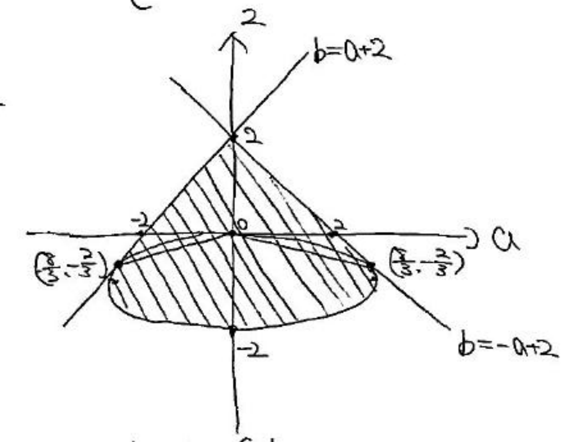
改めて書くと

$b\leqq -\frac{a}{4}$ かつ $b\leqq\frac{a}{4}$ のとき

$\begin{cases} b\leqq a+2 \\ \text{かつ} \\ b\leqq -a+2 \\ \text{かつ} \\ \frac{a^2}{8}+(b+1)^2\leqq 1 \end{cases}$

$b>-\frac{a}{4}$ または $b>\frac{a}{4}$ のとき

$\begin{cases} b\leqq a+2 \\ \text{かつ} \\ b\leqq -a+2 \end{cases}$



領域は斜線部分.
境界線含む.

(2) $\frac{b+1}{a+4}=k$ とおく.

$b+1=k(a+4)\cdots☆$

より $(-4,-1)$ を通る傾き k の直線

を D と共有するように動かす.

kが最大をとるとき ℓが $(0,2)$ を通るので

$$\max k = \frac{3}{4}$$

kが最小をとるとき ℓが楕円と接するときである.

$$\frac{a^2}{8} + k^2(a+4)^2 = 1$$

$$\Leftrightarrow (1+8k^2)a^2 + 64k^2a + 128k^2 - 8 = 0$$

$$\frac{D}{4} = 32^2k^4 - (1+8k^2)(128k^2-8)$$

$$= -128k^2 + 8 + 64k^2$$

$$= 8 - 64k^2 = 0$$

$$\therefore k^2 = \frac{1}{8}$$

$$\therefore \min k = -\frac{1}{2\sqrt{2}}$$

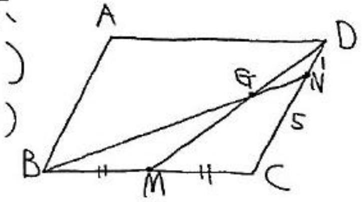
以上より

$$-\frac{1}{2\sqrt{2}} \leq \frac{b+1}{a+4} \leq \frac{3}{4}$$

4.

(1)

(i)



メネラウスの定理より

$$DG : GM = 2 : 5$$

$$\overrightarrow{AG}$$

$$= \overrightarrow{AD} + \overrightarrow{DG}$$

$$= \overrightarrow{AD} + \frac{2}{7}\overrightarrow{DM}$$

$$= \overrightarrow{AD} + \frac{2}{7}(\overrightarrow{DC} + \overrightarrow{CM})$$

（$\overrightarrow{CM} = -\frac{1}{2}\overrightarrow{AD}$）

$$= \frac{2}{7}\overrightarrow{AB} + \frac{6}{7}\overrightarrow{AD}$$

(ii) $A(x_a, y_a, z_a)$ とおくと

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$\Leftrightarrow \begin{pmatrix} 1-x_a \\ -y_a \\ -z_a \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{3} \\ -d \end{pmatrix}$$

$$A(1, -\sqrt{3}, d).$$

$$\overrightarrow{OG} = \frac{5}{7}\overrightarrow{OD} + \frac{2}{7}\overrightarrow{OM} = \begin{pmatrix} \frac{1}{7} \\ \frac{\sqrt{3}}{7} \\ \frac{5}{7}d \end{pmatrix}$$

$$\therefore \overrightarrow{AG} = \begin{pmatrix} -\frac{6}{7} \\ \frac{8}{7}\sqrt{3} \\ -\frac{2}{7}d \end{pmatrix}. \quad \overrightarrow{AD} = \begin{pmatrix} -1 \\ \sqrt{3} \\ 0 \end{pmatrix} \text{ より}$$

$$\overrightarrow{AG} \cdot \overrightarrow{AD} = |\overrightarrow{AG}||\overrightarrow{AD}|\cos\frac{\pi}{6}$$

$$\Leftrightarrow \frac{30}{7} = |\overrightarrow{AG}| 2 \cdot \frac{\sqrt{3}}{2}$$

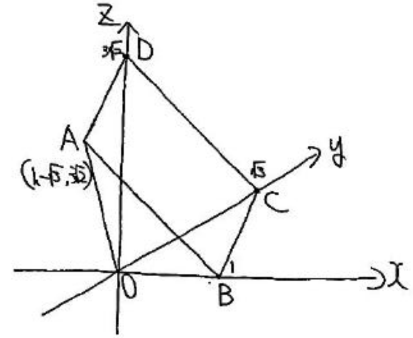
$$\Leftrightarrow \frac{30}{7} = \sqrt{3}\sqrt{\frac{36}{49} + \frac{192}{49} + \frac{4d^2}{49}}$$

↓ 7倍して2乗

$$900 = 3(228 + 4d^2)$$

$$\therefore d^2 = 18 \quad \therefore d = 3\sqrt{2} \quad (\because d > 0)$$

$$\therefore A(1, -\sqrt{3}, 3\sqrt{2})$$



線分 OA, AB, DC のどれが z軸から距離があるかを調べる.

$$OA : \begin{cases} x = t_1 \\ y = -\sqrt{3}t_1 \\ z = 3\sqrt{2}t_1 \end{cases} \quad (t_1 \in \mathbb{R})$$

$z = k$ のとき

$$x = \frac{k}{3\sqrt{2}}, \quad y = -\frac{\sqrt{3}k}{3\sqrt{2}} = -\frac{k}{\sqrt{6}}$$

$$\therefore x^2 + y^2 = \frac{k^2}{18} + \frac{k^2}{6} = \frac{2}{9}k^2 = p(k) \text{ とおく}$$

$$AB : \begin{cases} x = 1 \\ y = -\sqrt{3}t_2 \\ z = 3\sqrt{2}t_2 \end{cases} \quad (t_2 \in \mathbb{R})$$

$z = k$ のとき

$$x = 1, \quad y = -\frac{\sqrt{3}k}{3\sqrt{2}} = -\frac{k}{\sqrt{6}}$$

$$\therefore x^2 + y^2 = 1 + \frac{k^2}{6} = q(k) \text{ とおく.}$$

$$DC : \begin{cases} x = 0 \\ y = \sqrt{3} + \sqrt{3}t_3 \\ z = -3\sqrt{2}t_3 \end{cases} \quad (t_3 \in \mathbb{R})$$

$z = k$ のとき

$$x = 0, \quad y = \sqrt{3} - \frac{\sqrt{3}t}{3\sqrt{2}} = \sqrt{3} - \frac{k}{\sqrt{6}}$$

$$\therefore x^2 + y^2 = \left(\sqrt{3} - \frac{k}{\sqrt{6}}\right)^2 = r(k) \text{ とおく}$$

まず

$$p(k) - q(k) = \frac{k^2}{18} - 1 \leqq 0$$

$$\therefore p(k) \leqq q(k) \quad (\because 0 \leqq k \leqq 3\sqrt{2})$$

そして

$$q(k) - r(k) = 1 + \frac{k^2}{6} - \left(3 - 2\sqrt{\tfrac{1}{2}}k + \frac{k^2}{6}\right)$$

$$= \sqrt{2}k - 2$$

つまり以上より

$k \geqq \sqrt{2}$ のとき $q(k)$ が一番大きい.

$k < \sqrt{2}$ のとき $r(k)$ が一番大きい.

求める体積は

$$\int_{\sqrt{2}}^{3\sqrt{2}} q(k)\pi dk + \int_0^{\sqrt{2}} r(k)\pi dk$$

$$= \pi\left[t + \frac{t^3}{18}\right]_{\sqrt{2}}^{3\sqrt{2}} + \pi\left[-\frac{\sqrt{6}}{3}\left(\sqrt{3} - \frac{k}{\sqrt{6}}\right)^3\right]_0^{\sqrt{2}}$$

$$= \pi\left(6\sqrt{2} - \sqrt{2} - \frac{\sqrt{2}}{9}\right)$$

$$+ \pi\left\{-\frac{\sqrt{6}}{3}\left(\frac{2}{\sqrt{3}}\right)^3 + \frac{\sqrt{6}}{3} 3\sqrt{3}\right\}$$

$$= 7\sqrt{2}\pi$$