杏林(医) 2013

I

(a)

$$-x^2+4x-3\geqq 0$$

$$\therefore\ \underline{1\leqq x\leqq 3}\ /\!/$$

$$y=2\sqrt{-(x-2)^2+1}+1$$

$$\therefore\ \underline{1\leqq y\leqq 3}\ /\!/$$

(b)

$$y-1=2\sqrt{-(x-2)^2+1}$$

↓ 2乗して÷4

$$\frac{(y-1)^2}{4}=-(x-2)^2+1$$

$$\Leftrightarrow\ (x-2)^2+\frac{(y-1)^2}{4}=1$$

$$\Leftrightarrow\ X^2+\frac{Y^2}{4}=1$$

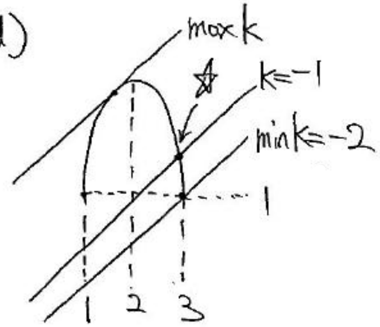
焦点 $(\underset{x-2}{X},\underset{y-1}{Y})=(0,\pm\sqrt{3})$

$$\Leftrightarrow\ (x,y)=\underline{(2,1\pm\sqrt{3})}\ /\!/ \leftarrow A,B$$

とすると

$$PA+PB=2b=4\ /\!/$$

(c)(d)



連立

$$(x-2)^2+\frac{(x+k-1)^2}{4}=1$$

$$\Leftrightarrow\ 4x^2-16x+16+x^2+2(k-1)x+(k-1)^2=4$$

$$\Leftrightarrow\ 5x^2+(2k-18)x+k^2-2k+13=0 \quad \cdots①$$

$$\frac{D}{4}=(k-9)^2-5(k^2-2k+13)$$

$$=-4k^2-8k+16=0$$

$$\Leftrightarrow\ k^2+2k-4=0$$

$$\therefore\ k=-1\pm\sqrt{5}$$

図より

$$\underline{-2\leqq k\leqq -1+\sqrt{5}}\ /\!/$$

☆のx座標は①で$k=-1$を代入

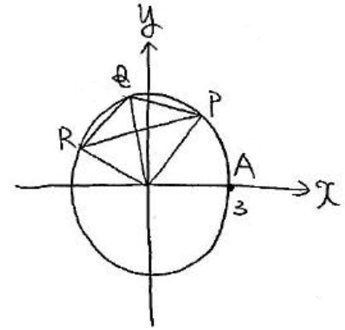
$$5x^2-20x+16=0$$

$$\therefore\ x=\frac{10\pm\sqrt{20}}{5}=2\pm\frac{2}{5}\sqrt{5}$$

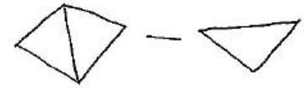
図より $\underline{1\leqq x\leqq 2+\frac{2}{5}\sqrt{5}}\ /\!/$

II

(a)



S

= 

$$=\frac{1}{2}\cdot 3\cdot 3\sin t\cdot 2 - \frac{1}{2}\cdot 3\cdot 3\sin 2t$$

$$=\underline{9\sin t-\frac{9}{2}\sin 2t}\ /\!/$$

$$\frac{dS}{dt}=9\cos t-9\cos 2t=0$$

$$\Leftrightarrow\ \cos t-2\cos^2 t+1=0$$

$$\Leftrightarrow\ 2\cos^2 t-\cos t-1=0$$

$$\Leftrightarrow\ \cos t=1,-\frac{1}{2}$$

t	$0\cdots\frac{2}{3}\pi\cdots\pi$
$\frac{dS}{dt}$	$+\ \ 0\ \ -$
S	↗ ↘

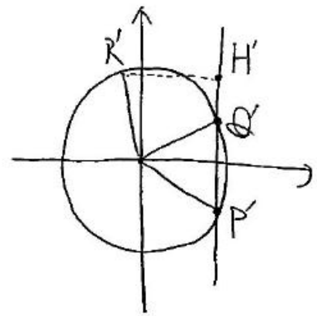
$t=\frac{2}{3}\pi$ のとき $\max S=\underline{\frac{27}{4}\sqrt{3}}\ /\!/$

(b)

$$\begin{pmatrix}\cos\frac{3}{2}t & \sin\frac{3}{2}t\\ \sin\frac{3}{2}t & \cos\frac{3}{2}t\end{pmatrix}$$

$$=\begin{pmatrix}\cos(-\frac{3}{2}t) & -\sin(\frac{3}{2}t)\\ \sin(-\frac{3}{2}t) & \cos(-\frac{3}{2}t)\end{pmatrix}$$

これは$-\frac{3}{2}t$回転の行列より



図のように全体を移動すれば

$$\underline{H'\left(3\cos\frac{1}{2}t,\ 3\sin\frac{3}{2}t\right)}\ /\!/$$

$$OH^2=OH'^2$$

$$=9\cos^2\frac{1}{2}t+9\sin^2\frac{3}{2}t$$

$$=9\,\frac{1+\cos t}{2}+9\,\frac{1-\cos 3t}{2}$$

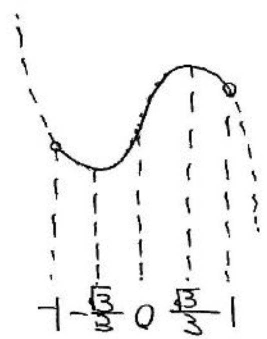
$$=9+\frac{9}{2}(\cos t-\underbrace{\cos 3t}_{4\cos^3 t-3\cos t})$$

$$=9+\frac{9}{2}(-4\cos^3 t+4\cos t)$$

$$=-18\cos^3 t+18\cos t+9$$

$$=-18u^3+18u+9=f(u)$$

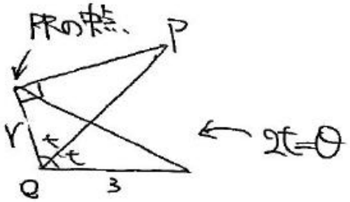
$f'(u) = -18(3u^2-1)$



$u = \cos t = \frac{\sqrt{3}}{3}$ のとき

$\max f(u) = \underline{9+4\sqrt{3}}$

(c)



$\cos t = \frac{r}{3}$

$\Leftrightarrow r = 3\cos t = \underline{3\cos\frac{1}{2}\theta}$

III.

(a) $\int_1^e \frac{1}{t}(\log t)^2 dt$

$= \left[\frac{1}{3}(\log t)^3\right]_1^e$

$= \underline{\frac{1}{3}}$

$f(x) = (\log x)^2 - \underbrace{\int_1^e \frac{f(t)}{t} dt}_{a\text{とおく}}$

$= (\log x)^2 - a$

$a = \int_1^e \left[\frac{(\log t)^2 - a}{t}\right] dt$

$= \frac{1}{3} - a\left[\log t\right]_1^e$

$\therefore a = \frac{1}{6}$, $\underline{f(x) = (\log x)^2 - \frac{1}{6}}$

$f'(x) = 2(\log x)\frac{1}{x}$

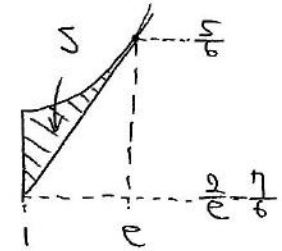
$f''(x) = \frac{2}{x^2} - \frac{2\log x}{x^2}$

$f''(e) = 0$ より 変曲点 $\underline{f(e) = \frac{5}{6}}$

(b) $(e, f(e))$ での接線は

$y = \frac{2}{e}(x-e) + \frac{5}{6}$

$= \underline{2e^{-1}x - \frac{7}{6}}$



S

$= \int_1^e \left\{(\log x)^2 - \frac{1}{6} - \left(\frac{2}{e}x - \frac{7}{6}\right)\right\} dx$

$= \int_1^e \left\{(\log x)^2 - \frac{2}{e}x + 1\right\} dx$

$= \left[x(\log x)^2 - 2x\log x + 2x - \frac{x^2}{e} + x\right]_1^e$

$= \left[x(\log x)^2 - 2x\log x + 3x - \frac{x^2}{e}\right]_1^e$

$= e - 2e + 3e - e - 3 + \frac{1}{e}$

$= e + \frac{1}{e} - 3 = \underline{-3 + \frac{1}{e}(1+e^2)}$

IV

$\vec{b}_{n+1} = \begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix}$

$= \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} + \begin{pmatrix} 2^n \log_2 \frac{(n+1)^2}{n} \\ 0 \end{pmatrix}$

$= \begin{pmatrix} a a_{n+1} + b a_n + 2^n \log_2 \frac{(n+1)^2}{n} \\ c a_{n+1} \end{pmatrix}$

上式より $a=3, b=-2, c=1$

$\therefore \underline{A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}}$

$(A^{-1})^{n-1}\vec{b}_n$ が階差型より オ…$\underline{n-1}$

(b) $\vec{b}_1 = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = \underline{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

$\underline{A^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}}$

対角化で表れるのは A^{-1} の固有値

$\varphi(t) = |A^{-1} - tE|$

$= \begin{vmatrix} -t & 1 \\ -\frac{1}{2} & \frac{3}{2}-t \end{vmatrix}$

$= t^2 - \frac{3}{2}t + \frac{1}{2} = 0$

$\therefore t = \frac{1}{2}, 1$

$\therefore \underline{A^{-1} = P\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}P^{-1}}$

$\sum_{k=1}^{n-1} (A^{-1})^k \begin{pmatrix} 2^k \log_2 \frac{(k+1)^2}{k} \\ 0 \end{pmatrix}$

$= \sum_{k=1}^{n-1} P(A^{-1})^k P^{-1} \begin{pmatrix} 2^k \log_2 \frac{(k+1)^2}{k} \\ 0 \end{pmatrix}$

$= \sum_{k=1}^{n-1} P\begin{pmatrix} \frac{1}{2^k} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2^k \log_2 \frac{(k+1)^2}{k} \\ 0 \end{pmatrix}$

$= \sum_{k=1}^{n-1} P\begin{pmatrix} \frac{1}{2^k} & -\frac{1}{2^k} \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2^k \log_2 \frac{(k+1)^2}{k} \\ 0 \end{pmatrix}$

$= \sum_{k=1}^{n-1} P\begin{pmatrix} \log_2 \frac{(k+1)^2}{k} \\ -2^k \log_2 \frac{(k+1)^2}{k} \end{pmatrix}$

ここで

$$\sum_{k=1}^{n-1} \log_2 \frac{(k+1)^2}{k}$$

$$= 2\log_2 n! - \log_2 (n-1)!$$

$$= \log_2 (n \cdot n!)$$

$$\sum_{k=1}^{n-1} 2^k \log_2 \frac{(k+1)^2}{k}$$

$$= \sum_{k=1}^{n-1} 2^k \{2\log_2(k+1) - \log_2 k\}$$

$$= \sum_{k=1}^{n-1} \{2^{k+1}\log_2(k+1) - 2^k \log_2 k\}$$

$$= 2^n \log_2 n$$

よって

(※※)の右辺の和の項は

$$P\begin{pmatrix} \log_2(n \times n!) \\ -2^n \log_2 n \end{pmatrix} \begin{matrix} \cdots \underline{0} \\ \cdots \underline{2} \end{matrix}$$

(c)

$$\varphi(t) = |A - tE|$$

$$= \begin{vmatrix} 3-t & -2 \\ 1 & -t \end{vmatrix}$$

$$= t^2 - 3t + 2 = 0$$

$\therefore t = 1, 2$

$$A = P\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$$

(※※)より両辺A^{n-1}をかけて

$$\vec{b_n} = A^{n-1}\vec{b_1} + A^{n-1}P\begin{pmatrix} \log_2(n \times n!) \\ -2^n \log_2 n \end{pmatrix}$$

$$= \begin{pmatrix} 2^n - 1 & -2^n + 2 \\ 2^{n-1} - 1 & -2^{n-1} + 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \log_2(n \times n!) \\ -2^n \log_2 n \end{pmatrix}$$

$$= \begin{pmatrix} 2^n & 1 \\ 2^{n-1} & 1 \end{pmatrix}\begin{pmatrix} \log_2(n \times n!) \\ -2^n \log_2 n \end{pmatrix}$$

$$\downarrow$$

$$a_n = 2^{n-1}\log_2(n \times n!) - 2^n \log_2 n$$

$$= 2^{n-1}\{\log_2(n \times n!) - \log_2 n^2\}$$

$$= 2^{n-1}\log_2 (n-1)!$$

$\cdots$ ①, ⑥

ちなみに

$$A^n = \{P\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}\}^n$$

$$= P\begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \cdots = \begin{pmatrix} 2^{n+1} - 1 & -2^{n+1} + 2 \\ 2^n - 1 & -2^n + 2 \end{pmatrix}$$