2012 横浜市立大医

[I]

(1)

$$f'(x)=\frac{\frac{1}{2}(a^2+x^2)^{-\frac{1}{2}}2x-1}{\sqrt{a^2+x^2}-x}$$

$$=\frac{x-\sqrt{a^2+x^2}}{(\sqrt{a^2+x^2}-x)\sqrt{a^2+x^2}}$$

$$=-\frac{1}{\sqrt{a^2+x^2}}$$

$$f''(x)=\frac{1}{2}(a^2+x^2)^{-\frac{3}{2}}2x$$

$$=x(a^2+x^2)^{-\frac{3}{2}}$$

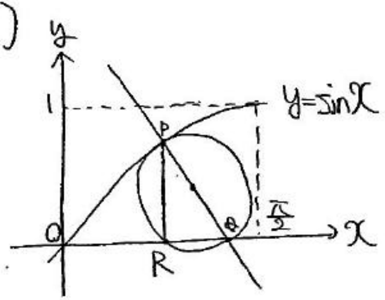
$$f'''(x)=(a^2+x^2)^{-\frac{3}{2}}-\frac{3}{2}(a^2+x^2)^{-\frac{5}{2}}2x$$

$$=(a^2+x^2)^{-\frac{3}{2}}-3x(a^2+x^2)^{-\frac{5}{2}}$$

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3$$

$$=\underline{\log a-\frac{x}{a}+\frac{x^3}{6a^3}}$$

(2)



法線: $y=-\frac{1}{\cos a}(x-a)+\sin a$

$$=-\frac{x}{\cos a}+\frac{a+\sin a\cos a}{\cos a}$$

$\therefore Q(a+\sin a\cos a,\ 0)$

$$S(a)=\sin a\cos a\times\sin a\times\frac{1}{2}$$

$$=\underline{\frac{1}{2}\sin^2 a\cos a}$$

$$=\frac{1}{2}(-\cos^3 a+\cos a)$$

$$=\frac{1}{2}(-t^3+t)\quad \lrcorner\ t=\cos a$$

$$\frac{dS(a)}{dt}=\frac{1}{2}(-3t^2+1)$$

t	0 ⋯ $\frac{1}{\sqrt{3}}$ ⋯ 1
$\frac{dS(a)}{dt}$	× + 0 − ×
$S(a)$	↗ ↘

$$\max S(a)=\frac{1}{2}\left(-\frac{1}{3\sqrt{3}}+\frac{3}{3\sqrt{3}}\right)=\underline{\frac{1}{3\sqrt{3}}}$$

(3) 第m群にあるとする

$$\left|\frac{1}{m},\frac{2}{m-1},\cdots,\frac{q}{p},\cdots\frac{m}{1}\right|$$

$$\downarrow$$

$$m-p+1=q$$

$$\therefore m=p+q-1$$

$\frac{q}{p}$は

$$\frac{1}{2}(m-1)m+q$$

$$=\underline{\frac{1}{2}(p+q-2)(p+q-1)+q\text{ 項}}$$

a_{100}が第m群にあるとすると

$$\frac{1}{2}(m-1)m<100\leqq\frac{1}{2}m(m+1)$$

$$\Leftrightarrow (m-1)m<200\leqq m(m+1)$$

$$m=14$$

$$\downarrow$$

$$91<100\leqq105$$

$$a_{100}=\underline{\frac{9}{6}}$$

(4)

$$A\begin{pmatrix}3&1\\2&1\end{pmatrix}=\begin{pmatrix}1&3\\1&2\end{pmatrix}$$

$$A=\begin{pmatrix}1&3\\1&2\end{pmatrix}\begin{pmatrix}1&-1\\-2&3\end{pmatrix}$$

$$=\begin{pmatrix}-5&8\\-3&5\end{pmatrix}$$

$$A\begin{pmatrix}5\\3\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$$

$$A\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$$

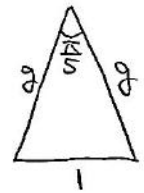
以上より

(i) nが奇数のとき $\begin{pmatrix}-1\\0\end{pmatrix}$

(ii) nが偶数のとき $\begin{pmatrix}5\\3\end{pmatrix}$

(5)

黄金三角形



$g=$(黄金数)

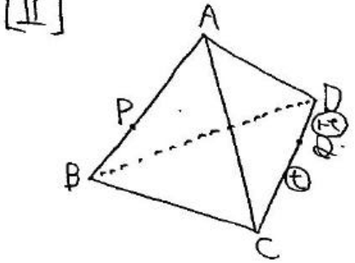
$$=\frac{1+\sqrt{5}}{2}$$

$$\sin\frac{\pi}{10}=\frac{\frac{1}{2}}{g}$$

$$=\frac{1}{1+\sqrt{5}}$$

$$=\underline{\frac{\sqrt{5}-1}{4}}$$

[II]



(1) $\overrightarrow{BA}\cdot\overrightarrow{BQ}$

$= \overrightarrow{BA}\cdot\{(1-t)\overrightarrow{BC}+t\overrightarrow{BD}\}$

$= (1-t)a\cdot a\cos 60^\circ + t\,a\cdot a\cos 60^\circ$

$= \dfrac{a^2}{2}$ //

(2) $\overrightarrow{QA}\cdot\overrightarrow{QB}$

$= \overrightarrow{AQ}\cdot\overrightarrow{BQ}$

$= \{(1-t)\overrightarrow{AC}+t\overrightarrow{AD}\}\cdot\{(1-t)\overrightarrow{BC}+t\overrightarrow{BD}\}$

$= (1-t)^2\dfrac{a^2}{2}+t^2\dfrac{a^2}{2}$

$(\because \overrightarrow{AC}\perp\overrightarrow{BD},\ \overrightarrow{AD}\perp\overrightarrow{BC})$

$= \dfrac{2t^2-2t+1}{2}a^2$ //

(3) $|\overrightarrow{QP}|^2$

$= |\overrightarrow{BP}-\overrightarrow{BQ}|^2$

$= |(1-t)\overrightarrow{BA}-\overrightarrow{BQ}|^2$

$= (1-t)^2a^2-2(1-t)\dfrac{a^2}{2}+|\overrightarrow{BQ}|^2$

$= (1-t)a^2(1-t-1)+|\overrightarrow{BQ}|^2$

$= -t(1-t)a^2+(1-t)^2a^2+2(1-t)t\dfrac{a^2}{2}+t^2a^2$

$= (2t^2-2t+1)a^2$

$\therefore |\overrightarrow{QP}| = \sqrt{2t^2-2t+1}\,a$ //

[III]

(1)

(i) $\alpha=\beta$ のとき

(与式)

$= \lim_{R\to\infty}\int_0^R \dfrac{1}{(1+\alpha x)^2}dx$

$= \lim_{R\to\infty}\left[-\dfrac{1}{\alpha}\cdot\dfrac{1}{1+\alpha x}\right]_0^R$

$= \lim_{R\to\infty}\left(-\dfrac{1}{\alpha}\cdot\dfrac{1}{1+\alpha R}+\dfrac{1}{\alpha}\right)=\dfrac{1}{\alpha}$ //

(ii) $\alpha\neq\beta$ のとき

$\dfrac{1}{(1+\alpha x)(1+\beta x)}=\dfrac{p}{1+\alpha x}+\dfrac{q}{1+\beta x}$

$=\dfrac{(p\beta+q\alpha)x+p+q}{(1+\alpha x)(1+\beta x)}$

$\begin{cases}\beta p+\alpha q=0\\ p+q=1\end{cases}$

解くと

$p=\dfrac{\alpha}{\alpha-\beta},\quad q=\dfrac{-\beta}{\alpha-\beta}.$

(与式)

$= \lim_{R\to\infty}\dfrac{1}{\alpha-\beta}\int_0^R\left(\dfrac{\alpha}{1+\alpha x}-\dfrac{\beta}{1+\beta x}\right)dx$

$= \lim_{R\to\infty}\dfrac{1}{\alpha-\beta}\Big[\log|1+\alpha x|-\log|1+\beta x|\Big]_0^R$

$= \lim_{R\to\infty}\dfrac{1}{\alpha-\beta}\log\dfrac{1+\alpha R}{1+\beta R}$

$= \lim_{R\to\infty}\dfrac{1}{\alpha-\beta}\log\dfrac{\frac{1}{R}+\alpha}{\frac{1}{R}+\beta}$

$= \dfrac{\log\alpha-\log\beta}{\alpha-\beta}$ //

(2) (1)の結果を用いると

(与式)

$= \int_0^\infty\left\{\dfrac{1}{a-b}\left(\dfrac{a}{1+ax}-\dfrac{b}{1+bx}\right)\dfrac{1}{1+cx}\right\}dx$

$= \dfrac{a}{a-b}\int_0^\infty\dfrac{1}{1+ax}\cdot\dfrac{1}{1+cx}dx$

$-\dfrac{b}{a-b}\int_0^\infty\dfrac{1}{1+bx}\cdot\dfrac{1}{1+cx}dx$

$= \dfrac{a}{a-b}\cdot\dfrac{\log a-\log c}{a-c}-\dfrac{b}{a-b}\cdot\dfrac{\log b-\log c}{b-c}$

$= \underset{A}{\dfrac{a}{(a-b)(a-c)}\log a}+\underset{B}{\dfrac{b}{(b-a)(b-c)}\log b}+\underset{C}{\dfrac{c}{(c-a)(c-b)}\log c}$ // // //

対称性を使って楽に