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[I]

(1)

$f(x)=x\sin^2 x$

$f'(x)=\sin^2 x+2x\sin x\cos x$

$=\sin x(\sin x+2x\cos x)$

x	0	$\cdots$	α	$\cdots$	π
$f'(x)$	0	$+$	0	$-$	0
$f(x)$		$\nearrow$		$\searrow$	

$f'(\alpha)=0 \iff \sin\alpha+2\alpha\cos\alpha=0$

$\iff \tan\alpha=-2\alpha$

$f(\alpha)=\alpha\sin^2\alpha$

$=\alpha\tan^2\alpha\cos^2\alpha$

$=4\alpha^3\dfrac{1}{1+\tan^2\alpha}$

$=\dfrac{4\alpha^3}{1+4\alpha^2}$ //

(2)

$a^4-2(b^2+c^2)a^2+b^4+c^4-2b^2c^2$

$=a^4-2(b^2+c^2)a^2+(b^2-c^2)^2$

$=a^4-2(b^2+c^2)a^2+(b+c)^2(b-c)^2$

$=\{a^2-(b+c)^2\}\{a^2-(b-c)^2\}$

$=(a+b+c)(a-b-c)$

$(a+b-c)(a-b+c)$ //

(3)

$(f*_t g)(x)$

$=\sum_{k=0}^{n}\dfrac{t^k}{2^k k!}(2a)^k e^{2ax}{}_nP_k x^{n-k}$

$=e^{2ax}\sum_{k=0}^{n}{}_nC_k(at)^k x^{n-k}$

$=e^{2ax}(at+x)^n$ //

(4)

$f(x)+e^{x}\int_0^x f(t)e^{-t}dt=\sin x$

$\downarrow x$で微分

$f'(x)+e^{x}\int_0^x f(t)e^{-t}dt$

$+f(x)=\cos x$

$\iff f'(x)+\sin x=\cos x$

$\iff f'(x)=\cos x-\sin x$

$\therefore f(x)=\sin x+\cos x+C$

$f(0)=0$ より

$f(x)=\sin x+\cos x-1$ //

[II]

(1)

$E-A=\begin{pmatrix}\frac{1}{3}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{3}\end{pmatrix}$

$(E-A)^{-1}$

$=\dfrac{1}{\frac{1}{9}+\frac{1}{4}}\begin{pmatrix}\frac{1}{3}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{3}\end{pmatrix}$

$=\dfrac{36}{13}\begin{pmatrix}\frac{1}{3}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{3}\end{pmatrix}$

$=\begin{pmatrix}\frac{12}{13}&-\frac{18}{13}\\\frac{18}{13}&\frac{12}{13}\end{pmatrix}$ //

(2)

$(X\ Y)=\left\{A\begin{pmatrix}x\\y\end{pmatrix}\right\}^t$

$=\begin{pmatrix}x\\y\end{pmatrix}^t A^t$

$=(x\ y)A^t$

X^2+Y^2

$=(X\ Y)\begin{pmatrix}X\\Y\end{pmatrix}$

$=(x\ y)A^tA\begin{pmatrix}x\\y\end{pmatrix}$

$=(x\ y)\begin{pmatrix}\frac{2}{3}&\frac{1}{2}\\-\frac{1}{2}&\frac{2}{3}\end{pmatrix}\begin{pmatrix}\frac{2}{3}&-\frac{1}{2}\\\frac{1}{2}&\frac{2}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$=(x\ y)\begin{pmatrix}\frac{4}{9}+\frac{1}{4}&0\\0&\frac{4}{9}+\frac{1}{4}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$=\dfrac{25}{36}(x^2+y^2)$

$\therefore \sqrt{X^2+Y^2}=\dfrac{5}{6}\sqrt{x^2+y^2}$ //

(3)

$\begin{pmatrix}x_n\\y_n\end{pmatrix}=A\begin{pmatrix}x_{n-1}\\y_{n-1}\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}$

$-)\ \begin{pmatrix}x_n-\alpha\\y_n-\alpha\end{pmatrix}=A\begin{pmatrix}x_{n-1}-\alpha\\y_{n-1}-\beta\end{pmatrix}$

$\begin{pmatrix}\alpha\\\beta\end{pmatrix}=A\begin{pmatrix}\alpha\\\beta\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}$

$\iff (E-A)\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

$\iff \begin{pmatrix}\alpha\\\beta\end{pmatrix}=(E-A)^{-1}\begin{pmatrix}3\\2\end{pmatrix}$

$=\begin{pmatrix}0\\6\end{pmatrix}$

これより

$\begin{pmatrix}x_n\\y_n-6\end{pmatrix}=A\begin{pmatrix}x_{n-1}\\y_{n-1}-6\end{pmatrix}$

が成立. (2)より

$x_n^2+(y_n-6)^2$

$=\dfrac{25}{36}\{x_{n-1}^2+(y_{n-1}-6)^2\}$

$=\cdots$

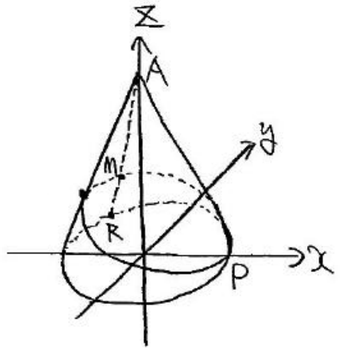
$=\left(\dfrac{25}{36}\right)^n\{x_0^2+(y_0-6)^2\}$

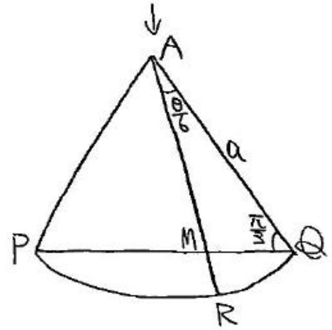
$\therefore \lim_{n\to\infty}\{x_n^2+(y_n-6)^2\}=0$

$\therefore \lim_{n\to\infty}x_n=0\quad \lim_{x\to\infty}y_n=6$ //

[Ⅲ]

(1)





弧PQ = Sの円周より

$2a \times \pi \times \frac{1}{6} = 2b\pi$

$\therefore b = \frac{a}{6}$

△AMQに正弦定理より

$\frac{AM}{\sin\frac{\pi}{3}} = \frac{a}{\sin\left(\frac{2}{3}\pi - \frac{\theta}{6}\right)}$

$\therefore AM = \frac{\sqrt{3}a}{2\sin\left(\frac{2}{3}\pi - \frac{\theta}{6}\right)}$

(2)

$\overrightarrow{AM} = \frac{AM}{a}\overrightarrow{AR}$

$= \frac{\sqrt{3}}{2\sin\left(\frac{2}{3}\pi - \frac{\theta}{6}\right)}\begin{pmatrix} b\cos\theta \\ b\sin\theta \\ -\sqrt{a^2 - \frac{a^2}{36}} \end{pmatrix}$

$\overrightarrow{OM}$

$= \overrightarrow{OA} + \overrightarrow{AM}$

$= \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{35}}{6}a \end{pmatrix} + \frac{\sqrt{3}}{2\sin\left(\frac{2}{3}\pi - \frac{\theta}{6}\right)}\begin{pmatrix} b\cos\theta \\ b\sin\theta \\ -\frac{\sqrt{35}}{6}a \end{pmatrix}$

($\overrightarrow{OM}$をxy平面に正射影したベクトル)

$= \frac{\sqrt{3}b}{2\sin\left(\frac{2}{3}\pi - \frac{\theta}{6}\right)}\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

$\therefore r = \frac{\sqrt{3}a}{12\sin\left(\frac{2}{3}\pi - \frac{\theta}{6}\right)}$

$\int_0^{2\pi} \frac{1}{2}\{r(\theta)\}^2 d\theta$

$= \frac{a^2}{96}\int_0^{2\pi} \frac{1}{\sin^2\left(\frac{2}{3}\pi - \frac{\theta}{6}\right)} d\theta$

$= \frac{a^2}{96}\int_0^{2\pi} \frac{1}{\cos^2\left\{\frac{\pi}{2} - \left(\frac{2}{3}\pi - \frac{\theta}{6}\right)\right\}} d\theta$

$= \frac{a^2}{96}\int_0^{2\pi} \frac{1}{\cos^2\left(\frac{\theta}{6} - \frac{\pi}{6}\right)} d\theta$

$= \frac{a^2}{96}\left[6\tan\left(\frac{\theta}{6} - \frac{\pi}{6}\right)\right]_0^{2\pi}$

$= \frac{a^2}{16}\left[\tan\frac{\pi}{6} - \tan\left(-\frac{\pi}{6}\right)\right]$

$= \frac{a^2}{8\sqrt{3}}$